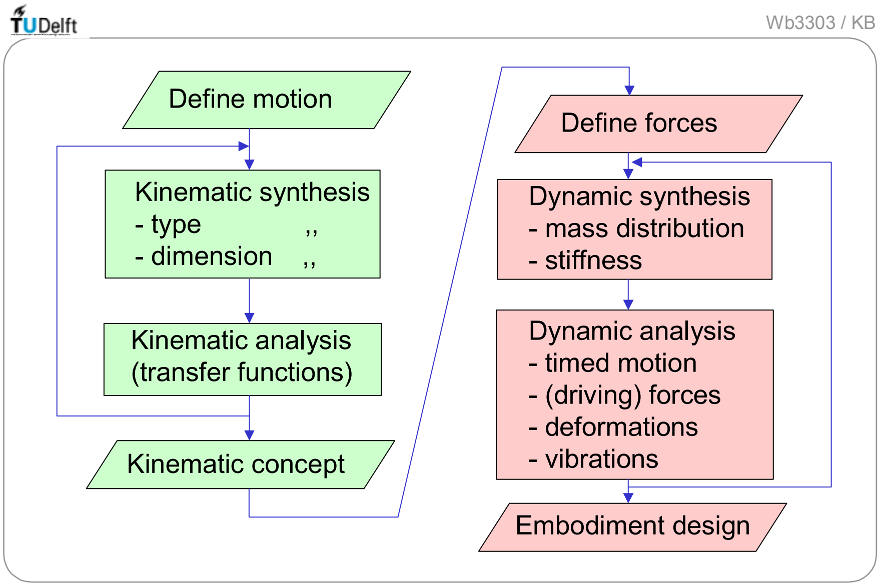

Fig 1.3.1 Kinematics and dynamics in the design process of mechanisms

5 Numerical kinematics with Finite Elements

5.1 Introduction

In the design process of mechanisms two major parts were distinguished: kinematics and dynamics, see again fig. 1.3.1.
The FEM (finite element method) has originally been developed for static and dynamic analysis of complex bar structures. Forces can be applied to such structures and the strength and deformations due to the forces can be calculated with the method. It is clear then that this method can be used in the dynamics part.
The use of FEM in kinematics allows a simplification, since forces are left out of consideration. Consequently deformations are not the result of forces. It is however possible to use the idea of a deformed link in kinematics. For instance: the specification that a link has a constant length can be expressed also with “the link has no elongation”. In this way deformations are used as kinematic constraints, with the intention to prescribe rigidity of a mechanical part. Deformation of the part is then just a geometrical change of the form, due to some mathematical variation process. The deformations can thus be considered as independent variables, which need to be prescribed.

A typical aspect of FEM is the *generalised approach* of the displacements (both translation and rotation) and the forces (both forces and moment of a force). Words like displacement, force and deformation will be regarded in this chapter as generalised quantities. Their physical dimension will appear from their application in the mechanical model.
It was observed that in the FEM-terminology there is not a generalised word for the length of a bar (although the word deformation is a generalisation of the word elongation, which means a change of the length). The word ***form parameter*** will be introduced here as a generalisation of length. Deformation is then a change of the form parameter.

Another aspect of the FEM is the powerful application to mechanical parts with a complex shape. Such a part can be divided into a great number of smaller elements with a simple nature (like simple bars). This approach gave the name to the method: *Finite* element method.
In kinematics the design of the mechanism is in a stage that the shape of a part is not yet under discussion. Just a few points are usually required to span the part or the plane. It is therefore not a good idea to think of such a division into a great number of smaller parts. Naturally the application of FEM in kinematics can be based on the assumption that ***each link can be described with one element.***

The major advantage of using the FEM for kinematics is that the set of ideas, terminology and mathematical rules can be used in the whole design process of mechanisms. Not only a reduction of methods to learn can be achieved, but also the understanding of the relation between kinematics and dynamics will be improved.
But in itself the FEM offers also the framework to do the numerical kinematics in a simple and attractive way, which certainly goes far beyond the possibilities of the graphical methods.
The explanation starts with the basic terms and relations, at hand of one simple element: the binary element. In chapter 5.3 a mechanism will be modelled with this element and the fundamental kinematics (transfer functions) are discovered in the FEM. A limited set of element types will be presented in chapter 5.4, with the aim to cover the wide variety of mechanism types. In numerical calculation it is usually required to give attention to calculation impossibilities and exception handling. In chapter 5.5 this will be done in relation to movability and transfer quality of the mechanism. In chapter 5.6 a brief description of a computer program will be given, which contains the theory presented so far in this chapter.

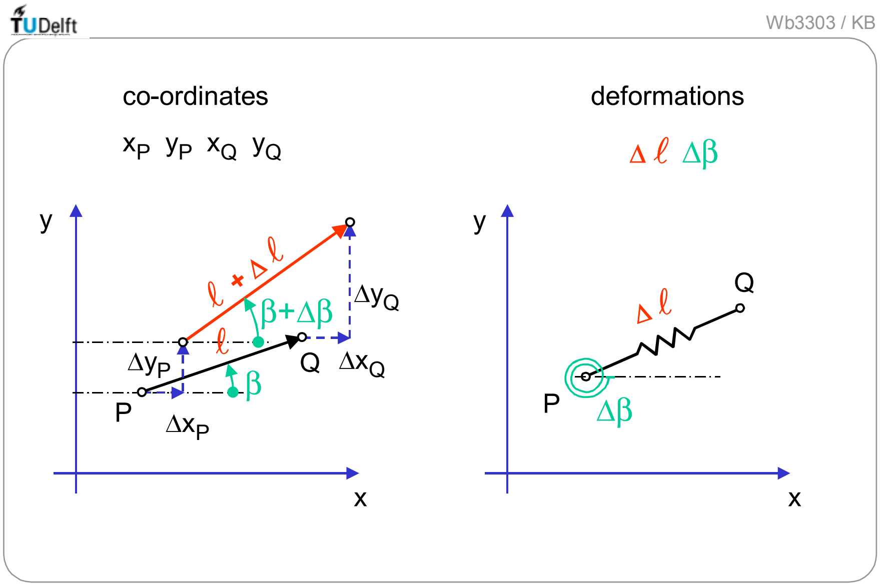

Fig. 5.2.1 Planar truss element, co-ordinates and deformations

5.2 Co-ordinates, deformations and continuity equations

Co-ordinates represent a moving part, plane or element. Frequently it concerns the co-ordinates of a point, but in the generalised approach of the FEM occasionally other types of co-ordinates are possible as well. It can be stated:

Each element is completely determined by its co-ordinates. Their values specify the position of the element (in the global co-ordinate system, connected to the frame). The motion of the element is fully determined by the motion of the co-ordinates.

The planar binary element will be referred to as an example, see fig. 5.2.1. Here the x- and y value of both end-points P and Q serve as the co-ordinates. They are contained in a vector $\underline{x}^k$, in which the superscript k indicates that this element has number k. Notation, using superscript T to indicate the transposed vector:

$$\underline{x}^k = \left| x_P \quad y_P \quad x_Q \quad y_Q \right|^T \tag{5.1}$$

The co-ordinates should be allowed to change independently of each other (mathematically independent variables). The actual form parameters are dependent on the co-ordinates of that element. One could speak of deformation when a certain form parameter has a prescribed value. In that case the prescribed form parameter contains a condition between the co-ordinates.

The (kinematic) conditions between the co-ordinates of an element are defined with the help of (kinematic) deformations. These deformations can be expressed as: actual value minus prescribed value of the form parameter.

For the binary element, two deformation types can be defined, see fig. 5.2.1 right part:

elongation $\Delta P = \ell - \ell^p$, and

rotation $\Delta\beta = \beta - \beta^p$

The actual values of ℓ and β are functions of the co-ordinates of the binary element:

$$f: \ \underline{x}^k \to \ell \qquad \ell = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2} \tag{5.2}$$

$$f: \ \underline{x}^k \to \beta \qquad \sin\beta = \frac{y_Q - y_P}{\ell} \quad \text{and} \quad \cos\beta = \frac{x_Q - x_P}{\ell}$$

$$\beta = \arctan\frac{y_Q - y_P}{x_Q - x_P} \tag{5.3}$$

Concerning the element k, the map of the vector space of co-ordinates upon the vector space of form parameters is called D^k. Applying the character ε for the form parameters, equations like (5.2) and (5.3) will in general be written as

$$D^k: \ \underline{x}^k \to \underline{\varepsilon}^k \tag{5.4}$$

in which for the binary element

$$\underline{\varepsilon}^k = \left| \ell \quad \beta \right|^T$$

The equation system (5.4) is called the ***continuity equation of the element (of order zero)***. Usually this is a non-linear equation system. In numerical methods like the FEM the linearisation is also required. This can be expressed with the derivative map

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$$x_Q - x_P = \ell.\cos\beta$$
$$y_Q - y_P = \ell.\sin\beta$$

$$D^k \bullet \Delta x^k = \Delta\varepsilon^k$$

$$\begin{vmatrix} -\cos\beta & -\sin\beta & \cos\beta & \sin\beta \\ \dfrac{\sin\beta}{\ell} & -\dfrac{\cos\beta}{\ell} & -\dfrac{\sin\beta}{\ell} & \dfrac{\cos\beta}{\ell} \end{vmatrix} \bullet \begin{vmatrix} \Delta x_P \\ \Delta y_P \\ \Delta x_Q \\ \Delta y_Q \end{vmatrix} = \begin{vmatrix} \Delta\ell \\ \Delta\beta \end{vmatrix}$$

$$D^k_{i,j} = \frac{\partial\varepsilon^k_i}{\partial x^k_j}$$

Fig. 5.2.2 First order relation between co-ordinates and deformations (continuity equations)

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element type	deformations	kinematic scheme
truss	$\Delta\ell = 0$ $\Delta\beta$ = free	
slider in frame	$\Delta\ell$ = free $\Delta\beta = 0$	
prismatic pair	$\Delta\ell$ = free $\Delta\beta$ = free	
actuator, translating	$\Delta\ell$ = prescribed $\neq 0$	
actuator, rotating	$\Delta\beta$ = prescribed $\neq 0$	

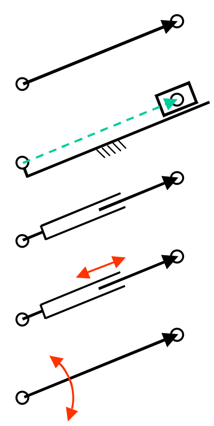

Fig. 5.2.3 Modelling kinematic structures with the binary FEM-element

$$\mathbf{D}D^k: \quad \Delta\underline{x}^k \rightarrow \Delta\underline{\varepsilon}^k \qquad \Delta\underline{\varepsilon}^k = \left[\frac{\partial\underline{\varepsilon}}{\partial\underline{x}}\right]^k \cdot \Delta\underline{x}^k = [D]^k \cdot \Delta\underline{x}^k \tag{5.5}$$

The matrix D^k, also known as the Jacobian matrix, contains the first order partial derivatives of the form parameters with respect to the co-ordinates. The matrix elements can be derived by partial differentiation of the continuity equations. For the binary element the result becomes, see also fig. 5.2.2:

$$\begin{vmatrix} \Delta\ell \\ \Delta\beta \end{vmatrix} = \begin{bmatrix} -\cos\beta & -\sin\beta & \cos\beta & \sin\beta \\ \frac{\sin\beta}{\ell} & -\frac{\cos\beta}{\ell} & -\frac{\sin\beta}{\ell} & \frac{\cos\beta}{\ell} \end{bmatrix} \times \begin{vmatrix} \Delta x_P \\ \Delta y_P \\ \Delta x_Q \\ \Delta y_Q \end{vmatrix} \tag{5.6}$$

and the matrix coefficients can be interpreted like

$$\frac{\partial\ell}{\partial x_P} = -\cos\beta \quad ; \quad \frac{\partial\beta}{\partial x_Q} = -\frac{\sin\beta}{\ell} \quad \text{etc.} \tag{5.7}$$

The matrix coefficients are of course functions of the co-ordinates. It will be preferred here to express them with terms like $\sin\beta$ and $\cos\beta$ (actually this expression has been derived earlier, see fig. 3.3.1). The terms of the zero order can be helpful in the first order!

The equation system (5.5) is also known as the ***continuity equation of first order***. It relates the changes between the co-ordinates and the form parameters. The (infinitesimal) variation of the form parameter $\Delta\varepsilon$ can be considered as deformation as well. Do not confuse it with the (finite) difference with the prescribed value of the form parameter.

The continuity equations are the basic equations to model elements. ***They offer the opportunity to give the form parameter a prescribed value or not***. This is demonstrated in figure 5.2.3 for the planar binary element. Typically this information about the element is to be given by a designer when specifying his kinematic model of the mechanism. In case of a prescribed form parameter an equation for the co-ordinates is then implicitly provided.

Remarks:

- Stating that a form parameter is constant (or having deformation zero) is one reason to prescribe the form parameter. A second reason can be that a certain form parameter must change to a new value. This option can be used to model input motion. Actuators can be given a non-constant (but prescribed) form parameter.
- An element, in the interpretation of the FEM here, does not need to be one part or plane. It is a good idea to consider an element as “just a group of co-ordinates” and possibly some relations.

5.3 Mechanisms and generalised transfer functions

5.3.1 The set of elements

A mechanism consists of a set of elements. The connections between the elements are defined by sharing co-ordinates. For instance: when two elements share the x- and y-co-ordinate of a node, then a (planar) revolute joint is specified between these two elements.

For the mechanism a co-ordinate space X can be defined, that consists of the union of co-ordinate spaces of the individual elements X^k:

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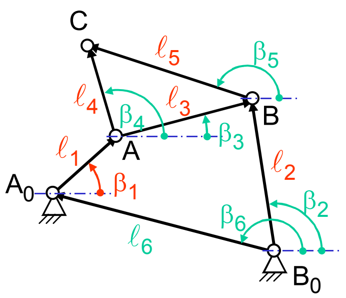

deformations $\Delta\varepsilon$	co-ordinates $\Delta\underline{x}$
$\Delta\ell_1 = 0$	$\Delta x_{Ao} = 0$
$\Delta\beta_1$= driving	$\Delta y_{Ao} = 0$
$\Delta\ell_2 = 0$	Δx_A = free
$\Delta\beta_2$= free	Δy_A = free
$\Delta\ell_3 = 0$	$\Delta x_{Bo} = 0$
$\Delta\beta_3$= free	$\Delta y_{Bo} = 0$
$\Delta\ell_4 = 0$	Δx_B = free
$\Delta\beta_4$= free	Δy_B = free
$\Delta\ell_5 = 0$	Δx_C = free
$\Delta\beta_5$= free	Δy_C = free
$\Delta\ell_6$ = free	
$\Delta\beta_6$= free	

Fig.5.3.1 Four-bar mechanism, modelled with binary elements

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$$\begin{array}{c}\\ \end{array}\quad x_A \quad y_A \quad x_B \quad y_B \quad x_C \quad y_C$$

$$\begin{vmatrix}\Delta\ell_1\\ \Delta\beta_1\\ \Delta\ell_2\\ \Delta\ell_3\\ \Delta\ell_4\\ \Delta\ell_5\end{vmatrix} = \begin{vmatrix} \cos\beta_1 & \sin\beta_1 & 0 & 0 & 0 & 0\\ -\dfrac{\sin\beta_1}{\ell_1} & \dfrac{\cos\beta_1}{\ell_1} & 0 & 0 & 0 & 0\\ 0 & 0 & \cos\beta_2 & \sin\beta_2 & 0 & 0\\ -\cos\beta_3 & -\sin\beta_3 & \cos\beta_3 & \sin\beta_3 & 0 & 0\\ -\cos\beta_4 & -\sin\beta_4 & 0 & 0 & \cos\beta_4 & \sin\beta_4\\ 0 & 0 & -\cos\beta_5 & -\sin\beta_5 & \cos\beta_5 & \cos\beta_5 \end{vmatrix} \times \begin{vmatrix}\Delta x_A\\ \Delta y_A\\ \Delta x_B\\ \Delta y_B\\ \Delta x_C\\ \Delta y_C\end{vmatrix} = \begin{vmatrix}0\\ \Delta\beta_1\\ 0\\ 0\\ 0\\ 0\end{vmatrix}$$

$$\mathbf{D^{pc}} \cdot \Delta\underline{\mathbf{x}}^{\mathbf{c}} = \Delta\underline{\varepsilon}^{\mathbf{p}}$$

Fig. 5.3.2 Continuity equations of the four-bar mechanism

$$\underline{x} \in X = \bigcup_k X^k \tag{5.8}$$

Concerning the (prescribed) form parameters: they can be applied to the individual elements as well as to the whole set of mechanism co-ordinates. In the latter case the relation concerns just a sub-set of the co-ordinates.
The list of the relations can be contained in a vector of (prescribed) form parameters $\underline{\varepsilon}$. The vector space E of these form parameters is then the direct sum of the element spaces E^k:

$$\underline{\varepsilon} \in E = \bigoplus_k E^k \tag{5.9}$$

Now the continuity equations of the whole mechanism can be defined as:

$$\begin{aligned} &D: \quad \underline{x} \rightarrow \underline{\varepsilon} \quad \underline{\varepsilon} = D(\underline{x}) \quad \text{and} \\ &\mathbf{D}D: \quad \Delta\underline{x} \rightarrow \Delta\underline{\varepsilon} \quad \Delta\underline{\varepsilon} = [D] \cdot \Delta\underline{x} \end{aligned} \tag{5.10}$$

Consider for instance the mechanism of figure 5.3.1 (left part): a planar four-bar linkage with coupler point C, which can be modelled with only binary elements.
The co-ordinate vector of this mechanism and the vector of form parameters are:

$$\underline{x} = \left| x_{A_0} \;\; y_{A_0} \;\; x_A \;\; y_A \;\; x_{B_0} \;\; y_{B_0} \;\; x_B \;\; y_B \;\; x_C \;\; y_C \right|^T \tag{5.11}$$

$$\underline{\varepsilon} = \left| \ell_1 \;\; \beta_1 \;\; \ell_2 \;\; \beta_2 \;\; \ell_3 \;\; \beta_3 \;\; \ell_4 \;\; \beta_4 \;\; \ell_5 \;\; \beta_5 \;\; \ell_6 \;\; \beta_6 \right|^T \tag{5.12}$$

To complete the definition of the mechanism the prescribed form parameters must be specified. Preparing for the next paragraph the co-ordinates will also be distinguished to be either fixed (prescribed) or moving (non-prescribed or "free"). With superscripts o or p for prescribed and c for "free" the vector spaces X and E are thus split up according:

$$\begin{aligned} &X = X^o \oplus X^c \quad \text{and} \\ &E = E^p \oplus E^c \end{aligned} \tag{5.13}$$

In the example this would give, see also figure 5.3.1 (right part):

$$\underline{x}^o = \left| x_{A_0} \;\; y_{A_0} \;\; x_{B_0} \;\; y_{B_0} \right|^T \quad \text{and} \quad \underline{x}^c = \left| x_A \;\; y_A \;\; x_B \;\; y_B \;\; x_C \;\; y_C \right|^T$$

$$\underline{\varepsilon}^p = \left| \ell_1 \;\; \beta_1 \;\; \ell_2 \;\; \ell_3 \;\; \ell_4 \;\; \ell_5 \right|^T \quad \text{and} \quad \underline{\varepsilon}^c = \left| \beta_2 \;\; \beta_3 \;\; \beta_4 \;\; \beta_5 \;\; \ell_6 \;\; \beta_6 \right|^T$$

Note that the element 6 is superfluous. Prescribing the length or the angle of this element would give a conflict with the fixed points A_0 and B_0. Such form parameters would introduce conflicting equations when prescribed.

5.3.2 First order continuity equations and transfer functions

Motion of the mechanism can be specified by the motion of the co-ordinates. With (5.8) this definition can be applied to either the co-ordinates of the individual elements or to the collected co-ordinates of the whole mechanism:

The (generalised) co-ordinates are the (generalised) transfer functions.

In the FEM-approach it concerns thus the map E→X, which is the reverse of the continuity equations. This reverse map can be defined more precisely with the help of (5.13). It will be clear that only the motion of the *moving* co-ordinates need to be calculated.

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$$
\begin{array}{cccccc} x_A & y_A & x_B & y_B & x_C & y_C \end{array}
$$

$$
\begin{vmatrix}
\cos\beta_1 & \sin\beta_1 & 0 & 0 & 0 & 0 \\
-\dfrac{\sin\beta_1}{\ell_1} & \dfrac{\cos\beta_1}{\ell_1} & 0 & 0 & 0 & 0 \\
0 & 0 & \cos\beta_2 & \sin\beta_2 & 0 & 0 \\
-\cos\beta_3 & -\sin\beta_3 & \cos\beta_3 & \sin\beta_3 & 0 & 0 \\
-\cos\beta_4 & -\sin\beta_4 & 0 & 0 & \cos\beta_4 & \sin\beta_4 \\
0 & 0 & -\cos\beta_5 & -\sin\beta_5 & \cos\beta_5 & \cos\beta_5
\end{vmatrix}
\times
\begin{vmatrix}
\dfrac{\partial x_A}{\partial \beta_1} \\
\dfrac{\partial y_A}{\partial \beta_1} \\
\dfrac{\partial x_B}{\partial \beta_1} \\
\dfrac{\partial y_B}{\partial \beta_1} \\
\dfrac{\partial x_C}{\partial \beta_1} \\
\dfrac{\partial y_C}{\partial \beta_1}
\end{vmatrix}
=
\begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}
$$

$$
D^{pc} \cdot \frac{\partial x^c}{\partial \beta_1} = \frac{\partial \varepsilon^p}{\partial \beta_1}
$$

1st order transfer function

Fig. 5.3.3 First order transfer functions can be calculated by solving the continuity equations (Matrix D^{pc} inversion)

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$$
\left[D^{pc}\right] \times
\begin{vmatrix}
\dfrac{\partial x_A}{\partial \ell_1} & \dfrac{\partial x_A}{\partial \beta_1} & \dfrac{\partial x_A}{\partial \ell_2} & \dfrac{\partial x_A}{\partial \ell_3} & \dfrac{\partial x_A}{\partial \ell_4} & \dfrac{\partial x_A}{\partial \ell_5} \\
\dfrac{\partial y_A}{\partial \ell_1} & \dfrac{\partial y_A}{\partial \beta_1} & \dfrac{\partial y_A}{\partial \ell_2} & \dfrac{\partial y_A}{\partial \ell_3} & \dfrac{\partial y_A}{\partial \ell_4} & \dfrac{\partial y_A}{\partial \ell_5} \\
\dfrac{\partial x_B}{\partial \ell_1} & \dfrac{\partial x_B}{\partial \beta_1} & \dfrac{\partial x_B}{\partial \ell_2} & \dfrac{\partial x_B}{\partial \ell_3} & \dfrac{\partial x_B}{\partial \ell_4} & \dfrac{\partial x_B}{\partial \ell_5} \\
\dfrac{\partial y_B}{\partial \ell_1} & \dfrac{\partial y_B}{\partial \beta_1} & \dfrac{\partial y_B}{\partial \ell_2} & \dfrac{\partial y_B}{\partial \ell_3} & \dfrac{\partial y_B}{\partial \ell_4} & \dfrac{\partial y_B}{\partial \ell_5} \\
\dfrac{\partial x_C}{\partial \ell_1} & \dfrac{\partial x_C}{\partial \beta_1} & \dfrac{\partial x_C}{\partial \ell_2} & \dfrac{\partial x_C}{\partial \ell_3} & \dfrac{\partial x_C}{\partial \ell_4} & \dfrac{\partial x_C}{\partial \ell_5} \\
\dfrac{\partial y_C}{\partial \ell_1} & \dfrac{\partial y_C}{\partial \beta_1} & \dfrac{\partial y_C}{\partial \ell_2} & \dfrac{\partial y_C}{\partial \ell_3} & \dfrac{\partial y_C}{\partial \ell_4} & \dfrac{\partial y_C}{\partial \ell_5}
\end{vmatrix}
= [1]
$$

$$
\left[\frac{\partial \varepsilon_i}{\partial x_j}\right] \times \left[\frac{\partial x_j}{\partial \varepsilon_k}\right] = [1]
$$

Fig. 5.3.4 All first order partial derivatives of the four-bar mechanism

Since the continuity equations are usually non-linear, the explanation starts with the first order continuity equations (5.10), but now split up according to (5.13).

$$\begin{vmatrix} \Delta\varepsilon^p \\ \Delta\varepsilon^c \end{vmatrix} = \begin{bmatrix} D^{pc} & D^{po} \\ D^{cc} & D^{co} \end{bmatrix} \times \begin{vmatrix} \Delta x^c \\ \Delta x^o \end{vmatrix} \quad (5.14)$$

The D-matrix is to be partitioned in accordance with (5.13). In a given position of the mechanism all coefficients of the D-matrix are given functions of the co-ordinates and they are thus calculable. Furthermore the fixed co-ordinates yield $\Delta x^o = 0$, so it follows

$$\begin{aligned} \Delta\varepsilon^p &= [D^{pc}] \cdot \Delta x^c \\ \Delta\varepsilon^c &= [D^{cc}] \cdot \Delta x^c \end{aligned} \quad (5.15)$$

Since $\Delta\varepsilon^p$ is given (prescribed, either zero or nonzero), the values of Δx^c can be calculated from the first equation of (5.15). It will be clear that matrix partition D^{pc} needs to be inverted. This means that this matrix must be ***square and regular***.

The example of the four-bar mechanism (fig. 5.3.1) can be considered further now. The first order continuity equations, concerning moving co-ordinates and prescribed form parameters, are written out in figure 5.3.2. Note that β_1 is the driving angle and that $\Delta\beta_1$ is considered as a small, but prescribed step of the input motion. To arrive at the definition of the first order transfer function the angle $\Delta\beta_1$ is taken now as an infinitesimal step (lim $\Delta\beta_1 \to 0$) after dividing all equations (all co-ordinates) by $\Delta\beta_1$, see figure 5.3.3.
In general the first order transfer function of moving co-ordinates $x^{/c}$ can be expressed as

$$\underline{x}^{/c} = [D^{pc}]^{-1} \cdot \underline{\varepsilon}^{/p} \quad (5.16)$$

in which the vector $\varepsilon^{/p}$ contains all zeroes except for a single 1. In other words:

The first order transfer function is a column vector of the inverse matrix D^{pc}. This column number is determined by the row number of the form parameter acting as input motion.

This rule certainly defines the transfer function of a single-degree-of-freedom mechanism. Before dealing with multi-DOF mechanisms the following consideration can be made.

Suppose another (prescribed) form parameter would be taken as input motion. In the example of the four-bar linkage (fig. 5.3.1) this could be one of the link lengths, while $\Delta\beta_1 = 0$. This change of input motion would not affect the D^{pc} matrix, the first order transfer function refers just to another column of the inverter D^{pc} matrix! Obviously it can be found:

The inverse matrix D^{pc} contains all partial derivatives of the (moving) co-ordinates with respect to the (prescribed) form parameters.

For the example this is depicted in figure 5.3.4.
The map $E^p \to X^c$ will be referred with italic capital character *F*, the derivative map **D***F* is linear and has a matrix noted with F^{cp}:

$$\mathbf{D}F: \quad \Delta\varepsilon^p \to \Delta x^c \qquad \left[F^{cp}\right] = \left[\frac{\partial \underline{x}^c}{\partial \underline{\varepsilon}^p}\right] = \left[D^{pc}\right]^{-1} \quad (5.16a)$$

In case of a multi-DOF mechanism the first order transfer function contains thus multi-columns of the inverse D^{pc} matrix.

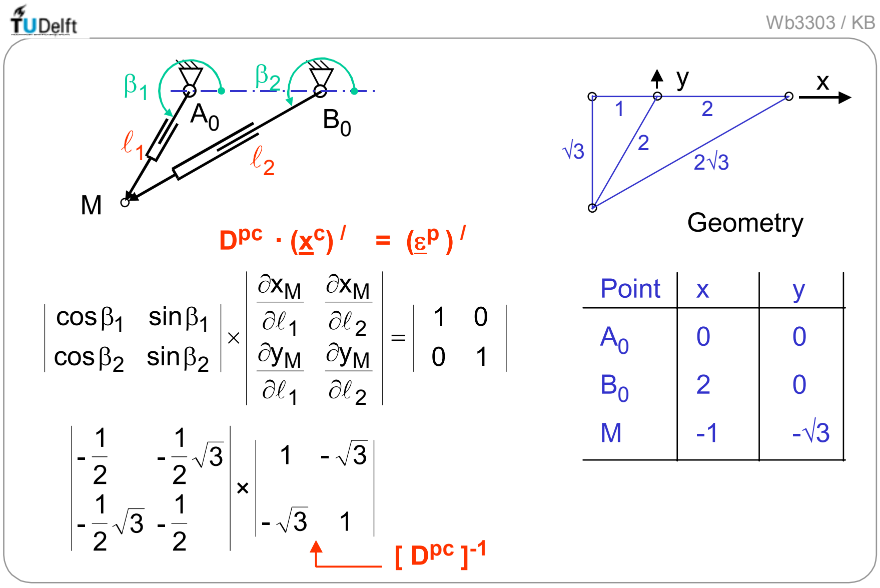

Fig. 5.3.5 Example of calculating the 1st order transfer functions (manipulator with 2 DOF)

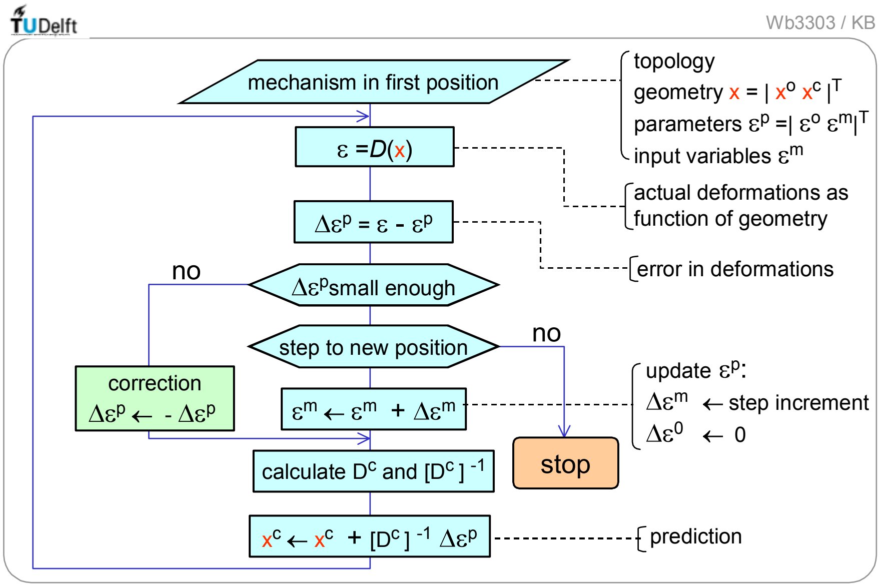

Fig.5.3.6 Basic iterative calculation scheme for a new position (zero order transfer function)

For convenience the prescribed form parameter vector ε^p can be split up further in “constant” (superscript o) and “driving” (superscript m, driving motor):

$$E = E^o \oplus E^m \oplus E^c \tag{5.17}$$

The part of matrix F^{cp} containing the first order transfer function can be denoted then as F^{cm}.

5.3.3 Example with a two-DOF mechanism

In figure 5.3.5 the mechanism has been depicted: a planar manipulator with two linear actuators. Such a mechanism can move point M across a certain region of the fixed plane. With the help of a FEM-model consisting of two binary links the first order information can be investigated easily. In any given position of the moving co-ordinates (here the x- and y-value of point M) the D^{pc}-matrix can be calculated and inverted. The inverted matrix contains all first order transfer functions of the mechanism. In this particular mechanism all (two) prescribed form parameters are inputs. So the F^{cp} matrix is identical to the first order transfer function matrix F^{cm} here.

5.3.4 The transfer function of order zero (position analysis)

To determine the position of a mechanism the values of the co-ordinates (x-vector) must be calculated. This concerns the non-linear map F: E→X. In the previous chapters it has been found that the first order map **D***F* (matrix D^{pc} concerning the linear equation system for continuity) is available in a given position of the mechanism. Based on this linearisation an iterative calculation scheme can be set up to calculate a new mechanism position, after having incremented the value of the degree(s) of freedom ε^m.

In figure 5.3.6 such an iterative scheme is depicted. In this picture matrix D^c should be understood as D^{pc} (in many publications the matrix has been noted also D^c) and underlining of vectors has been omitted.

The calculation starts with a given mechanism in an initial position (also called first position or reference position), that means:

- Given topology. All information of the elements like (shared) co-ordinates and (prescribed) form parameters is required. The vectors ε and x and the matrix D^c are then defined.
- Given co-ordinate vector x in the first mechanism position.

Theoretically the prescribed form parameters can be in conflict with the given co-ordinates, which would be called then a kinematically deformed mechanism. This can be detected in the first calculation block. Assume now these deformations are zero (or small enough) and a step to a new position is to be made. Those form parameters that act as a degree of freedom can be given a new value, but here only a relatively small increment $\Delta\varepsilon^m$ will be applied. With the vector of prescribed form parameters ε^p now changed the mechanism is “deformed”. The co-ordinate vector x^c should be updated then with an iterative scheme of prediction and correction (Newton/Rapson).

$$\text{Prediction: } x^c \leftarrow x^c + [D^c]^{-1} \cdot \Delta\varepsilon^p \tag{5.18}$$

$$\text{Correction: } \Delta\varepsilon^p \leftarrow -\Delta\varepsilon^p \tag{5.19}$$

The iteration cycle can be stopped when all kinematic deformations $\Delta\varepsilon^p$ are small enough. This decision is a responsibility of the user. The accuracy of the new position that can be achieved depends on the significance of the numbers, not on the algorithm. In the new position a second step can be made etc. The mechanism can move in this way from position to position.

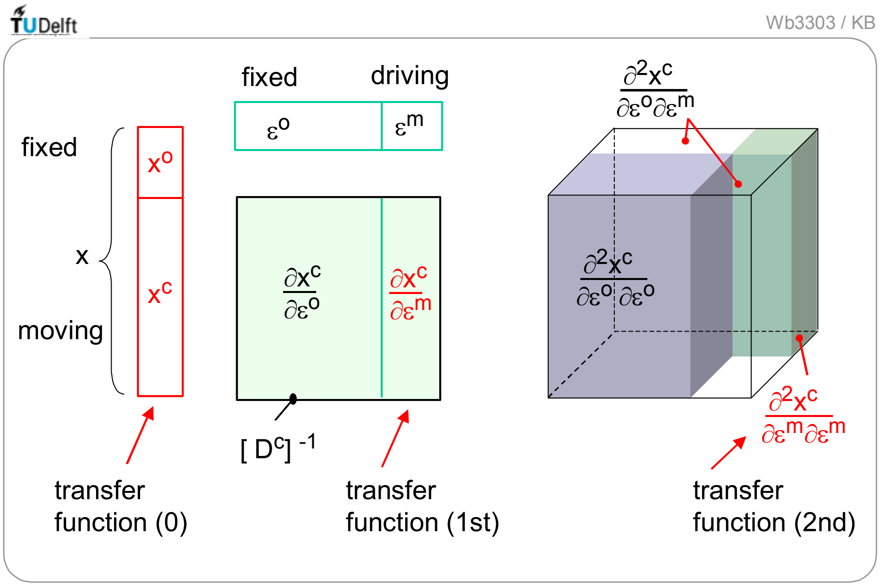

Fig. 5.3.7 Generalised transfer functions, FEM-definition (overview)

Remarks:

1) For big steps this algorithm could become divergent. The user is responsible to avoid this problem. For practical use the algorithm is robust enough. For instance: a crank can usually be given a full revolution in 24 steps. The problem can for instance arise when mistakes in the input values have been made.
2) The algorithm can handle also a mechanism that is (kinematically) deformed in the initial position. The moving co-ordinates could be adjusted then to the prescribed form parameters. This option is very powerful in practice: it allows specifying the co-ordinate vector with inaccurate (estimated) values. In many practical situations the link lengths are the quantities that are known to the user and they can be introduced in the calculation scheme (otherwise, when not explicitly specified, the prescribed value of a form parameter can be taken from the initially given co-ordinates).
3) Compared with direct algebraic formulas this numerical algorithm requires extra information, namely the initial position of the mechanism. This can however also be felt as an advantage, since the specified co-ordinates avoid ambiguities of the configuration (no cognate problems).

5.3.5 The transfer function of second order

Second order derivatives were not required in the iteration process to calculate a new mechanism position. They can be calculated afterwards when needed in a design problem. In the new position also the first order transfer function (actually all first order derivatives) can be (re-) calculated.

The first order continuity equations (5.16) need to be differentiated. They will be rewritten in index notation (matrix-vector notation is not very adequate for more indices):

$$\underline{\varepsilon}^{/p} = [D^c] \cdot \underline{x}^{/c} \tag{5.20}$$

In index notation:

$$\varepsilon_i^{/p} = D_{i,j}^c \cdot x_j^{/c} \tag{5.21}$$

In index notation a repeated subscript in a multiplication indicates also summation. So in this case $D_{i,j}^c \cdot x_j^{/c}$ should be read as $\sum_j D_{i,j}^c \cdot x_j^{/c}$. The repeated subscript j is used as a counter in this term and can be used also in another term as a counter. The comma between the subscripts indicates a differentiation operation. The advantage of the index notation is that further differentiation can be expressed simply by adding a further subscript to the term to be differentiated. Assuming a mechanism with one degree of freedom, differentiation of (5.21) with respect to this DOF yields:

$$0_i = D_{i,j}^c \cdot x_j^{//c} + D_{i,jk}^c \cdot x_j^{/c} \cdot x_k^{/c}$$

Writing the second order transfer function $x^{//c}$ explicitly:

$$x_j^{//c} = -F_{j,i}^c \cdot D_{i,kl}^c \cdot x_k^{/c} \cdot x_l^{/c} \tag{5.22}$$

For better understanding the derivatives can be written out like:

$$\frac{d^2 x_j^c}{d(\varepsilon^m)^2} = -\frac{\partial x_j^c}{\partial \varepsilon_i^p} \cdot \frac{\partial^2 \varepsilon_i^p}{\partial x_k^p \partial x_l^p} \cdot \frac{dx_k^c}{d\varepsilon^m} \cdot \frac{dx_l^c}{d\varepsilon^m}$$

The term $D^c_{i,kl}$ contains the second order derivations of the continuity matrix D^c. These terms are of course functions of the co-ordinates (of the element the form parameter belongs to).

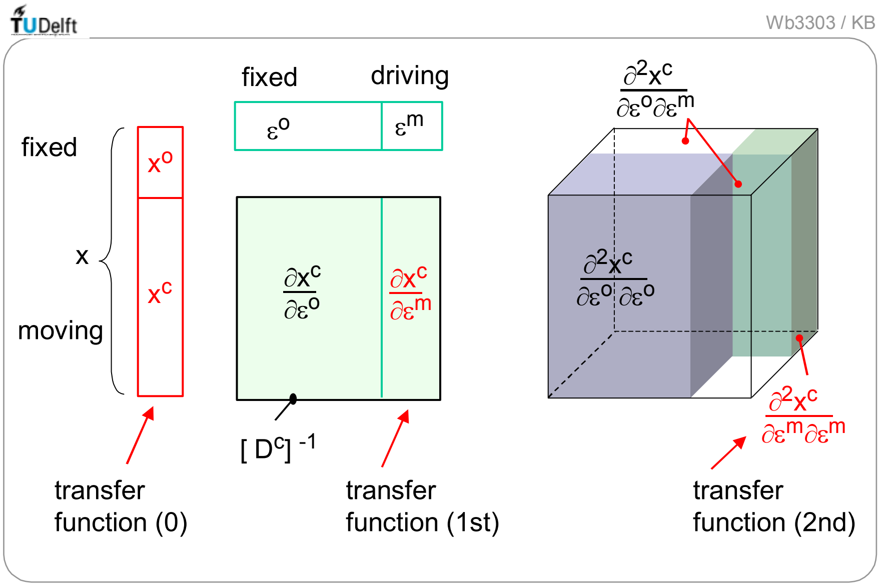

Fig. 5.3.7 Generalised transfer functions, FEM-definition (overview)

These functions need to be derived once for all elements. For each form parameter these second order derivatives concern a square, symmetrical matrix with the dimension of the number of co-ordinates of that element. For the binary element the two 4x4 matrices concerning the form parameters “length” and “angle” will be given as an example.

$$\frac{\partial^2 \ell}{\partial x_i \partial x_j} = \frac{1}{\ell} \begin{vmatrix} \sin^2\beta & -\sin\beta\cos\beta & -\sin^2\beta & \sin\beta\cos\beta \\ -\sin\beta\cos\beta & \cos^2\beta & \sin\beta\cos\beta & -\cos^2\beta \\ -\sin^2\beta & \sin\beta\cos\beta & \sin^2\beta & -\sin\beta\cos\beta \\ \sin\beta\cos\beta & -\cos^2\beta & -\sin\beta\cos\beta & \cos^2\beta \end{vmatrix} \tag{5.23}$$

$$\frac{\partial^2 \beta}{\partial x_i \partial x_j} = \frac{1}{\ell^2} \begin{vmatrix} \sin 2\beta & -\cos 2\beta & -\sin 2\beta & \cos 2\beta \\ -\cos 2\beta & -\sin 2\beta & \cos 2\beta & \sin 2\beta \\ -\sin 2\beta & \cos 2\beta & \sin 2\beta & -\cos 2\beta \\ \cos 2\beta & \sin 2\beta & -\cos 2\beta & -\sin 2\beta \end{vmatrix} \tag{5.24}$$

Such matrix coefficients can be derived straightforward. For instance, with the help of (5.6/7):

$$\frac{\partial \ell}{\partial x_P} = -\cos\beta \qquad ; \frac{\partial^2 \ell}{\partial x_P \partial y_P} = \sin\beta \cdot \frac{\partial \beta}{\partial y_P} = -\frac{\sin\beta\cos\beta}{\ell} \qquad \text{etc.} \tag{5.25}$$

$$\frac{\partial \beta}{\partial x_P} = \frac{\sin\beta}{\ell} \qquad ; \frac{\partial^2 \beta}{\partial x_P \partial y_P} = \frac{1}{\ell^2}\left(\ell\cos\beta \frac{\partial \beta}{\partial y_P} - \sin\beta \frac{\partial \ell}{\partial y_P} \right) = -\frac{\cos 2\beta}{\ell^2} \qquad \text{etc.} \tag{5.26}$$

Remarks:

1) The term $D^c_{i,kl}$ in (5.22) can be understood as one big block, assembled from the elements, with the dimension of the total number of moving co-ordinates. This block must be multiplied then twice with the first order coordinate vector. The resulting product can also be achieved more efficiently by doing the multiplication “element after element”, and sum the results. This procedure saves memory space and avoids frequent multiplication with zero.
2) The transfer function of the second order concerns only a part of all second order derivatives. For (internal) further calculations, for instance in kinematic optimisation (see chapter 6) all second order terms $F^c_{i,jk}$ are required. They can be calculated in a comparable way, starting with the derivation of the first order continuity equation

$$\delta_{ik} = D^c_{i,j} \cdot F^c_{j,k} \tag{5.27}$$

in which δ_{ik} is the so called Kronecker Delta (the unity matrix). Differentiation with respect to all prescribed form parameters yields:

$$0 = D^c_{i,j} \cdot F^c_{j,kl} + D^c_{i,nm} \cdot F^c_{n,k} \cdot F^c_{m,l}$$

$$F^c_{j,kl} = -F^c_{j,i} \cdot D^c_{i,nm} \cdot F^c_{n,k} \cdot F^c_{m,l} \tag{5.28}$$

The complete overview of first and second order transfer functions, as can be calculated with the FEM theory, is depicted in fig. 5.3.7. It will be clear that these results include mechanisms with multiple degree of freedom.

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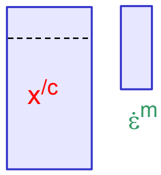
Map: T → E^m → X

f: $x(\varepsilon^m(t))$



$$\dot{x}^c = \frac{\partial x^c}{\partial \varepsilon^m}.\frac{d\varepsilon^m}{dt} = x^{/c}.\dot{\varepsilon}^m$$

$$\ddot{x}^c = \frac{\partial^2 x^c}{\partial \varepsilon^m \partial \varepsilon^m}\cdot\frac{d\varepsilon^m}{dt}\cdot\frac{d\varepsilon^m}{dt} + \frac{\partial x^c}{\partial \varepsilon^m}\cdot\frac{d^2\varepsilon^m}{dt^2} = \dot{\varepsilon}^{mT}.x^{//c}.\dot{\varepsilon}^m + x^{/c}.\ddot{\varepsilon}^m$$

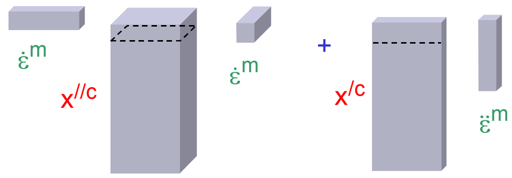


Fig. 5.3.8 Timed motion, multi degree of freedom

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$$\varepsilon^{/c} = \frac{\partial \varepsilon^c}{\partial \varepsilon^m} = \frac{\partial \varepsilon^c}{\partial x^k}.\frac{\partial x^k}{\partial \varepsilon^m} = [D^{ck}]\cdot x^{/k}$$

$$\dot{\varepsilon}^c = \frac{d\varepsilon^c}{dt} = \frac{\partial \varepsilon^c}{\partial x^k}.\frac{dx^k}{dt} = [D^{ck}]\cdot \dot{x}^k$$

$$\varepsilon^{//c} = \frac{\partial^2 \varepsilon^c}{\partial \varepsilon^m \partial \varepsilon^m} = \frac{\partial \varepsilon^c}{\partial x^k}\cdot\frac{\partial^2 x^k}{\partial \varepsilon^m \partial \varepsilon^m} + \frac{\partial x^k}{\partial \varepsilon^m}\cdot\frac{\partial^2 \varepsilon^c}{\partial x^k \partial x^k}\cdot\frac{\partial x^k}{\partial \varepsilon^m}$$

$$= [D^{ck}]\cdot x^{//k} + x^{/kT}\cdot[D^{ckk}]\cdot x^{/k}$$

$$\ddot{\varepsilon}^c = \frac{d^2\varepsilon^c}{dt^2} = [D^{ck}]\cdot\ddot{x}^k + \dot{x}^{kT}\cdot[D^{ckk}]\cdot\dot{x}^k$$

Fig. 5.3.9 Transfer functions and timed motion of dependent form parameters

5.3.6 Generalised timed motion

In addition to chapter 1.4 timed motion can be considered now in a generalised way. This means the velocities and the accelerations of all co-ordinates will be calculated now according (visualisation in a matrix and vector style see fig. 5.3.8):

$$\dot{x}^c = \frac{\partial x^c}{\partial \varepsilon^m} . \frac{d\varepsilon^m}{dt} = x^{/c} . \dot{\varepsilon}^m \tag{5.29}$$

$$\ddot{x}^c = \frac{\partial^2 x^c}{\partial \varepsilon^m \partial \varepsilon^m} . \frac{d\varepsilon^m}{dt} . \frac{d\varepsilon^m}{dt} + \frac{\partial x^c}{\partial \varepsilon^m} . \frac{d^2 \varepsilon^m}{dt^2} = \dot{\varepsilon}^{mT} . x^{//c} . \dot{\varepsilon}^m + x^{/c} . \ddot{\varepsilon}^m \tag{5.30}$$

This calculation is straightforward for given input velocity and acceleration vectors $\dot{\varepsilon}^m$ and $\ddot{\varepsilon}^m$. Note that the calculation is actually done "co-ordinate after co-ordinate". The equations (5.29) and (5.30) are thus also valid for the velocity and acceleration of one particular co-ordinate, once the transfer functions up to second order are known.

5.3.7 Transfer functions of dependent form parameters

Dependent (non-prescribed) form parameters are of course also functions of the degrees of freedom. They can also be understood as transfer functions and as functions of time. So a second type of generalised transfer functions will be introduced here.
Basically, by the definition of elements, they are determined by the motion of the co-ordinates x^k of the element they belong to. In a generalised way they can be expressed by the continuity equations of the element. With superscript c for the dependent form parameter and referring to (5.6) the expressions are:

$$\varepsilon^{/c} = \frac{\partial \varepsilon^c}{\partial \varepsilon^m} = \frac{\partial \varepsilon^c}{\partial x^k} \cdot \frac{\partial x^k}{\partial \varepsilon^m} = [D^{ck}] \cdot x^{/k} \tag{5.31}$$

$$\begin{aligned} \varepsilon^{//c} &= \frac{\partial^2 \varepsilon^c}{\partial \varepsilon^m \partial \varepsilon^m} = \frac{\partial \varepsilon^c}{\partial x^k} \cdot \frac{\partial^2 x^k}{\partial \varepsilon^m \partial \varepsilon^m} + \frac{\partial^2 \varepsilon^c}{\partial x^k \partial x^k} \cdot \frac{\partial x^k}{\partial \varepsilon^m} \cdot \frac{\partial x^k}{\partial \varepsilon^m} \\ &= [D^{ck}] \cdot x^{//k} \quad + \quad x^{/kT} \cdot [D^{ckk}] \cdot x^{/k} \end{aligned} \tag{5.32}$$

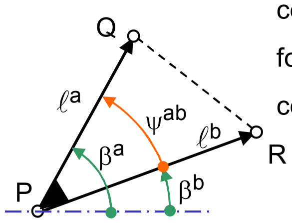
The term D^{ckk} refers to a second order continuity matrix like (5.23) or (5.24) in case of a binary element. The transfer functions of the co-ordinates (up to second order) of the element have already been calculated. With the help of the timed values (5.29/30) the timed dependent form parameters (velocities and accelerations) can be calculated as expressed in fig. 5.3.9.

5.4 Description of element types

5.4.1 General consideration

Up to now a general theory for kinematic analysis of mechanism has been described. The only element used so far is the planar truss element, which is however the basic element for bar linkages. It will be clear that with only this element not all mechanism types, as described in chapter 2, can be modelled. The purpose of this chapter will be then to present a set of other element types, which should be sufficient to cover all modelling problems of planar mechanisms.
It must be remarked that no unique receipt to model a mechanism can be given. The choice for a certain type of element can be influenced by for instance:

- Wish for a small number of co-ordinates (small mechanism model),
- Specific properties or (dis)advantages, easiness to understand etc.



co-ordinates $x^{kT} = | x_P \; y_P \; x_Q \; y_Q \; x_R \; y_R |$

form parameters $\varepsilon^{kT} = | \ell^a \; \ell^b \; \psi^{ab} |$

cont. equation $\psi^{ab} = \beta^a - \beta^b$

$\Delta\psi^{ab} = \Delta\beta^a - \Delta\beta^b$

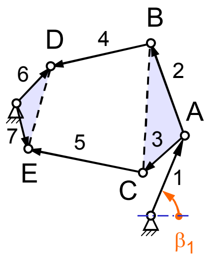
1st order continuity equation

$$\Delta\psi^{ab} = \left| \left(\frac{s^a}{\ell^a} - \frac{s^b}{\ell^b}\right) \; \left(-\frac{c^a}{\ell^a} + \frac{c^b}{\ell^b}\right) \; -\frac{s^a}{\ell^a} \; \frac{c^a}{\ell^a} \; \frac{s^b}{\ell^b} \; -\frac{c^b}{\ell^b} \right| \times \begin{vmatrix} \Delta x_P \\ \Delta y_P \\ \Delta x_Q \\ \Delta y_Q \\ \Delta x_R \\ \Delta y_R \end{vmatrix}$$

abbr: $s^a = \sin \beta^a$

$c^a = \cos \beta^a$, etc

Fig. 5.4.1 Ternary element: alternative definition with fixed angle (comparable with torsion element out of plane)



x_A	y_A	x_B	y_B	x_C	y_C	x_D	y_D	x_E	y_E	
c_1	s_1	0	0	0	0	0	0	0	0	ℓ_1
$-\frac{s_1}{\ell_1}$	$\frac{c_1}{\ell_1}$	0	0	0	0	0	0	0	0	β_1
$-c_2$	$-s_2$	c_2	s_2	0	0	0	0	0	0	ℓ_2
$-c_3$	$-s_3$	0	0	c_3	s_3	0	0	0	0	ℓ_3
$\frac{s_2}{\ell_2} - \frac{s_3}{\ell_3}$	$\frac{c_3}{\ell_3} - \frac{c_2}{\ell_2}$	$-\frac{s_2}{\ell_2}$	$\frac{c_2}{\ell_2}$	$\frac{s_3}{\ell_3}$	$-\frac{c_3}{\ell_3}$	0	0	0	0	$\beta_2 - \beta_3$
0	0	$-c_4$	$-s_4$	0	0	c_4	s_4	0	0	ℓ_4
0	0	0	0	$-c_5$	$-s_5$	0	0	c_5	s_5	ℓ_5
0	0	0	0	0	0	c_6	s_6	0	0	ℓ_6
0	0	0	0	0	0	$-\frac{s_6}{\ell_6}$	$\frac{c_6}{\ell_6}$	$\frac{s_7}{\ell_7}$	$-\frac{c_7}{\ell_7}$	$\beta_6 - \beta_7$
0	0	0	0	0	0	0	0	c_7	s_7	ℓ_7

Fig. 5.4.2 Example with ternary elements (fixed angle) Stephenson-2 mechanism

In general it can be stated that: the choice for an element type ***depends on the goal to be achieved by the user***. This rule will become even more manifest in dynamics.

In accordance with the general theory a lot of element types could be defined. The minimum requirement to define an element type is:

- The set of co-ordinates of the element must consist of independent variables with unlimited region (global co-ordinate space).
- The set of form parameters must contain continuous and differentiable (at least twice) functions of the co-ordinates of the element. Form parameters should preferably be clear geometric quantities, which allow large deformations. In that case they can serve as driving quantities.

It must be noticed that the binary element does not completely fulfil these requirements. The two points are not allowed to coincide, because the angle β will become undefined and thus coefficients of matrix D^c cannot be calculated. This element should only be used when the length is not close to zero. Such a situation could for instance occur when the length is not prescribed (slider). The binary element is however very understandable for the user and the disadvantage of prohibited zero length will in practice occur only in special situations, which might be recognised by the user.
Cartesian x- and y-values are very well understandable types of co-ordinates for the user. Some other co-ordinate types will however be introduced, but only when this is necessary or advantageous. To understand the element type it is sufficient to consider the continuity equations up to the first order.

To keep the amount of element types as less as possible the idea of **combining form parameters** will be adopted. It means that a new form parameter can be defined that is a linear combination of two (or more) other ones according to:

$$\varepsilon_{new} = f_1\varepsilon_1 + f_2\varepsilon_2 \qquad (5.33)$$

in which f_1 and f_2 are constant factors.
Adoption of this idea hardly introduces complications in the general theory. The required terms of the continuity equations, including the derivative terms, of the new form parameter can easily be constructed from the known expressions of the form parameters ε_1 and ε_2 to be combined. This idea will be used in the following paragraph.

5.4.2 Ternary element including a fixed angle

A ternary element can normally be defined with three binary elements. In special cases this modelling might introduce problems, for instance when the three nodes are in-line. With three prescribed lengths a dependency in the continuity equations will occur, causing then an irregular matrix D^c (inversion is impossible).
With an alternative modelling this problem can be avoided, see fig. 5.4.1. Two binary links a and b are used which include a fixed angle ψ^{ab}. The continuity equations of ψ^{ab} can be combined according (5.33) from the equations of the individual form parameters β^a and β^b, which were derived as (5.6). To create β^a - β^b the multiplication factors +1 (for β^a) and –1 (for β^b) must be used and the results must be added.

Obviously there is no need to define such a special ternary element type. The user can create it easily when needed.
It can be remarked that the length of a or b can either be kept constant (prescribed) or left free (dependent length, slider). One of the angles β^a or β^b can still be prescribed, for instance to specify input motion. Both angles can be used in other combined form parameters, as long as no dependencies will be introduced.

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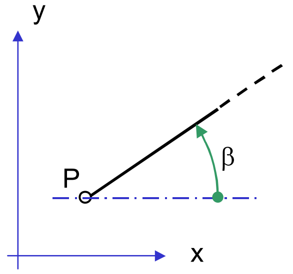

co-ordinates $x^{kT} = | x_P \; y_P \; \beta |$

form parameters $\varepsilon^{kT} = | \beta_v |$

cont. equation $\beta_v = \beta$

1st order continuity equation

$$\Delta\beta_v = \begin{vmatrix} 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} \Delta x_P \\ \Delta y_P \\ \beta \end{vmatrix}$$

Fig. 5.4.3 "Half beam" element: angle as a co-ordinate

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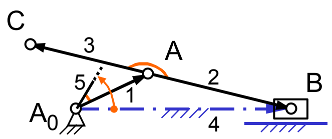

$\psi_{32} = \beta_3 - \beta_2$

$\psi_{51} = \beta_5 - \beta_1$

at one part

$$\Delta\varepsilon^P = \begin{vmatrix} \Delta\ell_1 \\ \Delta\psi_{51} \\ \Delta\beta_{5v} \\ \Delta\ell_2 \\ \Delta\ell_3 \\ \Delta\psi_{32} \\ \Delta\beta_4 \end{vmatrix} = \begin{vmatrix} c_1 & s_1 & 0 & 0 & 0 & 0 & 0 \\ \frac{s_1}{\ell_1} & -\frac{c_1}{\ell_1} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -c_2 & -s_2 & c_2 & s_2 & 0 & 0 & 0 \\ -c_3 & -s_3 & 0 & 0 & c_3 & s_3 & 0 \\ \frac{s_3}{\ell_3} - \frac{s_2}{\ell_2} & -\frac{c_3}{\ell_3} + \frac{c_2}{\ell_2} & \frac{s_2}{\ell_2} & -\frac{c_2}{\ell_2} & -\frac{s_3}{\ell_3} & \frac{c_3}{\ell_3} & 0 \\ 0 & 0 & -\frac{s_4}{\ell_4} & \frac{c_4}{\ell_4} & 0 & 0 & 0 \end{vmatrix} \times \begin{vmatrix} \Delta x_A \\ \Delta y_A \\ \Delta x_B \\ \Delta y_B \\ \Delta x_C \\ \Delta y_C \\ \Delta\beta_5 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ \Delta\beta_{5v} \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

Fig. 5.4.4 Slider-crank mechanism with a phase angle at the input link, modelled with a fixed angle to a "half beam"

The Stephenson-2 mechanism of fig. 5.4.2 has been modelled using only binary elements. The angles between links 2-3 and 6-7 respectively include a fixed angle. The matrix D^c shows how the angles β_2–β_3 and β_6–β_7 have been added with multiplication factors +1 and –1 .

5.4.3 Half beam element (angular co-ordinate)

Some parts (like wheels) can adequately be described with a point (the centre of the wheel) and a direction (orientation). A length parameter does not come into account then, but regarding the option of combining form parameters, the angle as a form parameter makes certainly sense. To fit into the FEM-theory the angle of the element should be represented by two quantities:

- As a co-ordinate (called β) and
- As a form parameter (called β_v).

Due to this double definition a continuity equation exists: β and β_v must be equal. Their difference specifies a deformation, which could for instance be interpreted as torsion (out of the plane). The formal description of this element is depicted in fig. 5.4.3. The name of the element refers to the fact that one point of the beam is left out consideration.

Except for wheels this element can be used to give any other (binary) element an angular reference line, which can be in a different direction as the link. A phase angle at a crank can be modelled in this way, see the example of figure 5.4.4. Here the element 5 is a half beam element, which is to be connected at a fixed angle ψ_{51} to element 1 (the crank, a binary element). The fixed angle can be created by combining the following two form parameters according (5.33):

- Form parameter β_{5v} of the half beam (multiplication factor +1) and
- Form parameter β_1 of the binary element (multiplication factor –1).

Because the angle of element 5 is taken as input angle, the form parameter β_{5v} itself must be prescribed also. Note that the elements 1 and 5 form one part and involve three continuity equations. The example shows also a fixed angle in the coupler plane, which consists of the in-line binary elements 2 and 3.

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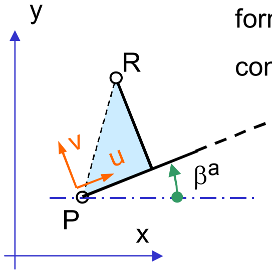

co-ordinates $x^{kT} = | x_P \; y_P \; \beta^a \; x_R \; y_R |$

form parameters $\varepsilon^{kT} = | \beta^a_v \; u \; v |$

cont. equations $\beta^a_v = \beta^a$

$$u = (x_R - x_P)\cos\beta^a + (y_R - y_P)\sin\beta^a$$

$$v = (x_P - x_R)\sin\beta^a + (y_R - y_P)\cos\beta^a$$

1st order continuity equations

$$\begin{vmatrix} \Delta\beta^a_v \\ \Delta u \\ \Delta v \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ -\cos\beta^a & -\sin\beta^a & v & \cos\beta^a & \sin\beta^a \\ \sin\beta^a & -\cos\beta^a & -u & -\sin\beta^a & \cos\beta^a \end{vmatrix} \times \begin{vmatrix} \Delta x_P \\ \Delta y_P \\ \Delta\beta^a \\ \Delta x_R \\ \Delta y_R \end{vmatrix}$$

Fig. 5.4.5 Ternary element (1): point R(u,v) added to a "half beam"

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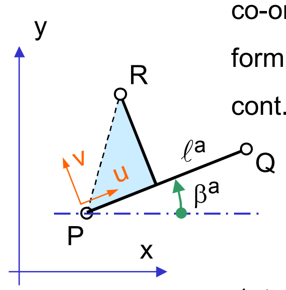

co-ordinates $x^{kT} = |\, x_P \; y_P \; x_Q \; y_Q \; x_R \; y_R \,|$

form parameters $\varepsilon^{kT} = |\, \ell^a \;\; \beta^a \;\; u \;\; v \,|$

cont. equations:
- ℓ^a and β^a *see binary element PQ*
- $u = (x_R - x_P)\cos\beta^a + (y_R - y_P)\sin\beta^a$
- $v = (x_P - x_R)\sin\beta^a + (y_R - y_P)\cos\beta^a$

1st order continuity equations

$$\begin{vmatrix} \Delta u \\ \Delta v \end{vmatrix} = \begin{vmatrix} (-c^a + \frac{v}{\ell^a} s^a) & (-s^a - \frac{v}{\ell^a} c^a) & -\frac{v}{\ell^a} s^a & \frac{v}{\ell^a} c^a & c^a & s^a \\ (s^a - \frac{u}{\ell^a} s^a) & (-c^a + \frac{u}{\ell^a} c^a) & \frac{u}{\ell^a} s^a & -\frac{u}{\ell^a} c^a & -s^a & c^a \end{vmatrix} \times \begin{vmatrix} \Delta x_P \\ \Delta y_P \\ \Delta x_Q \\ \Delta y_Q \\ \Delta x_R \\ \Delta y_R \end{vmatrix}$$

Fig. 5.4.6 Ternary element (2): point R(u,v) added to a binary element

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element type	deformations
part without slider	$\Delta u = 0$ $\Delta v = 0$
slider along bar	$\Delta u =$ free $\Delta v = 0$
slider perpendicular to bar	$\Delta u = 0$ $\Delta v =$ free
double slider (Scotch yoke)	$\Delta u =$ free $\Delta v =$ free

kinematic scheme

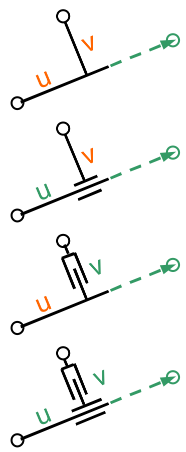

Fig. 5.4.7 Modelling sliders with the ternary elements (1) or (2)

5.4.4 Ternary elements (using local co-ordinates)

In paragraph 5.4.2 it was shown that a ternary element could easily be created with two binary elements and a fixed angle. In cases that one of the lengths (or both lengths) is zero or almost zero, an alternative modelling of the ternary element is required.

Such a ternary element could be based on the ***half beam element***. This element has namely the angle β as a co-ordinate (independent variable) and is thus defined in any situation. Relative to the local co-ordinate system (origin in x,y and abscissa in direction β, see fig. 5.4.5) the extra point R can be added using the local variables u and v. Here u and v, as local geometric quantities, should be considered as form parameters. The continuity equations concerning u and v (and β_v of the half beam) are depicted in the figure. The matrix coefficients for the D-matrix can be derived straightforward. Note that the values u and v (quantities of order zero) can be re-used in the first order terms, not only for abbreviation, but also for better understanding. It can be detected easily that when P and R coincide (u = v = 0), all matrix coefficients remain defined. This element will further be referred to as the Ternary-1 element (despite the missing third point).

The same idea of using local co-ordinates can be used with the ***binary element*** (Ternary-2 element). For many users this seems quit more understandable to apply than the ternary-1 element. The definition however has a complication, since angle β is now not available as a co-ordinate, but depends on the position of the points P and Q of the binary element (they are not allowed to coincide). Furthermore the derivation of the functions $u(x^k)$ and $v(x^k)$ with respect to all co-ordinates x^k has a complication: the chain rule must be applied for differentiation of β .The final result is depicted in figure 5.4.6. Compared with the ternary-1 element the matrix coefficients are more complicated (less understandable).

A favourable application of the local quantities u and v concerns the modelling of a slider. Either u or v (or both at a time) can be specified as free (dependent). The corresponding kinematic schemes are depicted in fig. 5.4.7. The ideas can be applied to both the Ternary-1 and the Ternary-2 element type.

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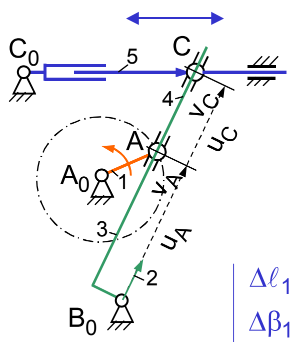

$$x^T = |\; x_{Ao}\; y_{Ao}\; x_A\; y_A\; x_{Bo}\; y_{Bo}\; x_C\; y_C\; \beta_2\; x_{Co}\; y_{Co} \;|$$

$$x^{cT} = |\; x_A\; y_A\; x_C\; y_C\; \beta_2 \;|$$

$$\varepsilon^T = |\; \ell_1\; \beta_1\; \beta_{2v}\; u_A\; v_A\; u_C\; v_C\; \ell_5\; \beta_5 \;|$$

binary "half" ternary ternary binary

$$\varepsilon^{pT} = |\; \ell_1\; \beta_1\; v_A\; v_C\; \beta_5 \;|$$

$$\Delta\varepsilon^p = \begin{vmatrix} \Delta\ell_1 \\ \Delta\beta_1 \\ \Delta v_A \\ \Delta v_C \\ \Delta\beta_5 \end{vmatrix} = \begin{vmatrix} c_1 & s_1 & 0 & 0 & 0 \\ -\frac{s_1}{\ell_1} & \frac{c_1}{\ell_1} & 0 & 0 & 0 \\ -s_2 & c_2 & 0 & 0 & -u_A \\ 0 & 0 & -s_2 & c_2 & -u_C \\ 0 & 0 & -\frac{s_5}{\ell_5} & \frac{c_5}{\ell_5} & 0 \end{vmatrix} \times \begin{vmatrix} \Delta x_A \\ \Delta y_A \\ \Delta x_C \\ \Delta y_C \\ \Delta\beta_2 \end{vmatrix} = \begin{vmatrix} 0 \\ \Delta\beta_1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

Fig. 5.4.8 Modelling sliders in a quick-return mechanism

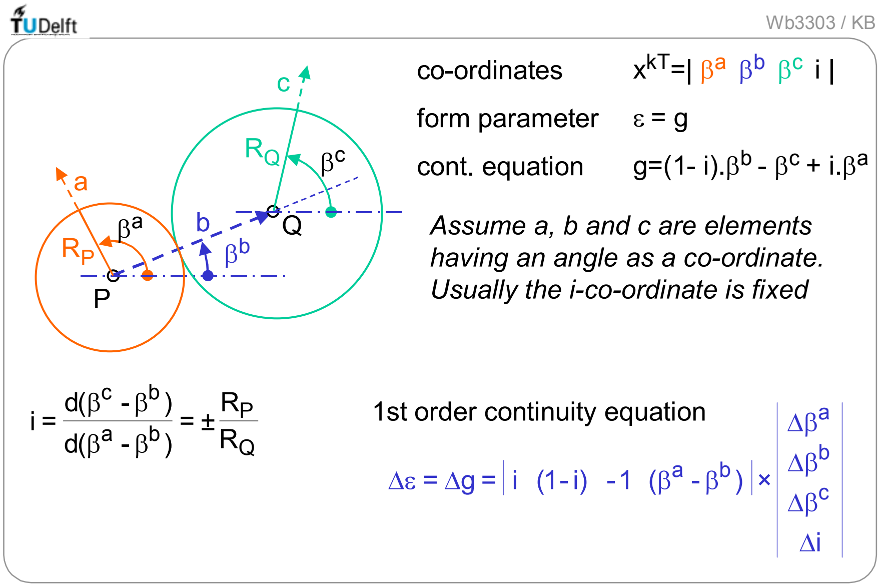

Fig.5.4.9 *Pair-of-gears element (1), using transmission coefficient i as a co-ordinate*

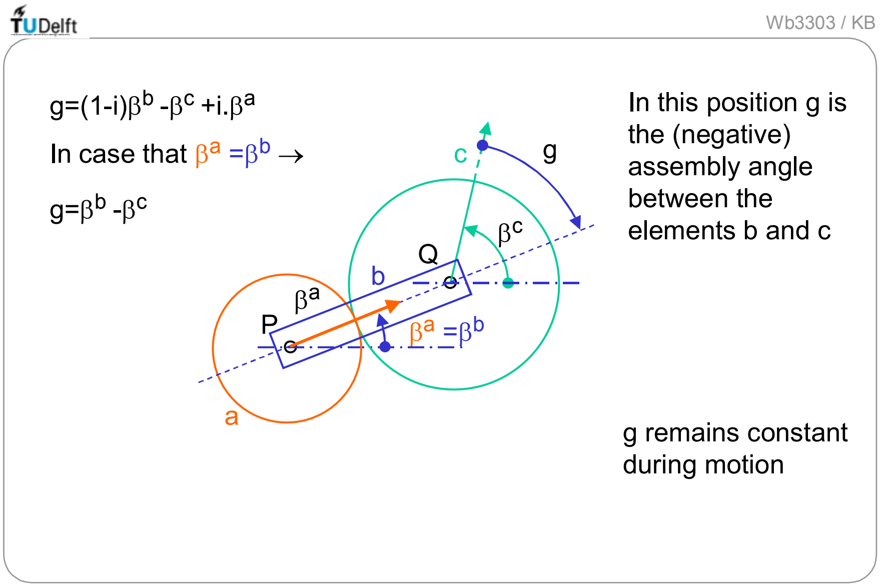

Fig. 5.4.10 *Interpretation of the form parameter g of a pair of gears as the assembly angle*

An example of a mechanism with several slider pairs is shown in fig. 5.4.8. The forward stroke of point C corresponds to a considerable longer part of the crank cycle than the return stroke. That's why this mechanism is also known as a "quick return mechanism". To model all parts of this mechanism five FEM-elements can be used:

- The driven crank A_0A is simply a binary element (element 1).
- Point C is for this occasion the end-point of a binary element (nr. 5), with free length and fixed horizontal direction. Length C_0C is assumed to be always (much) greater than zero.
- The slot guides two sliding points (A and C). Based on a half beam (element 2) the ternary 1 element is applied twice (nrs. 3 and 4) to define the extra points A and C. For both points the u-parameter (along the direction of element 2) has been specified as free.

Finally the D^c matrix has the dimension 5x5 as presented in the figure. As an exercise the reader might try to make a model of this mechanism with a 4x4 matrix.

5.4.5 Pair of gears

Actually a gear pair causes just a (constant) relation between rotation of moving parts. In the FEM-theory relations are used between the co-ordinates of elements. To model a gear pair becomes easy when the angles are defined as co-ordinates (this is not yet a new idea, in the previous paragraphs such elements were already used). A general model of a gear pair, including the connecting bar, is presented in fig. 5.4.9. The elements a and c can be any element with angular co-ordinate β^a and β^c respectively. Bar b should also have an angular co-ordinate. The ternary-1 element can be used here for instance (u = prescribed bar length, v = 0). The angular relation between the elements a, b and c can be written as:

$$i = \frac{d(\beta^c - \beta^b)}{d(\beta^a - \beta^b)} = \pm \frac{R_P}{R_Q} \tag{5.34}$$

The sign of i, the transmission constant, depends on the contact type: outer/outer contact shows a negative value and inner/outer contact a positive value for i. Normally the gear radii R_P and R_Q are constants and thus is i a constant. When the relation (5.34) is not fulfilled a continuity equation (form parameter) should express to what extent this happens. For this purpose eq. (5.34) can be written as

$$\Delta\varepsilon = (1 - i)\Delta\beta^b - \Delta\beta^c + i \cdot \Delta\beta^a = 0 \tag{5.35}$$

Here the first order continuity equation seems to be directly found. The original equation (zero order) could be found by integration of (5.35):

$$\varepsilon = (1 - i)\beta^b - \beta^c + i \cdot \beta^a \tag{5.36}$$

A point of attention is the role of the quantity i in the equation. Two different approaches can be considered now.

1. The user can take the initiative and compose the equation (5.36) by hand, referring to the composition rule (5.33). The constant values (1-i), -1 and +i must be applied then as multiplication factors.
2. The equation (5.36) is to be generated completely within the theory. In that case a method must be organised to pass the i-value to the equation. A general method therefore is to adopt the quantity i as a co-ordinate. This requires i to be an independent variable and there is no objection to do this. It can even be an advantage: a variable quantity i means that an adjustable variator can be modelled also (but normally i would be used as a fixed co-ordinate).

In case of a non-constant value of i the first order continuity equation (5.35) is not correct. The chain rule must be applied, and the result is finally presented in fig. 5.4.9. This element will be referred to further as the ***Gear-1 element***.

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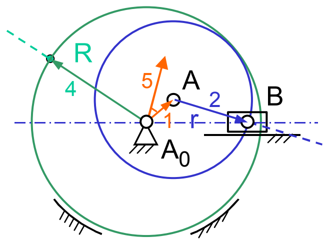
$x^T = | x_{Ao}\ y_{Ao}\ x_A\ y_A\ x_B\ y_B\ \beta_4\ \beta_5\ \beta_2\ i |$

$x^{cT} = | x_A\ y_A\ x_B\ \beta_2\ \beta_4\ \beta_5 |$

$\varepsilon^{pT} = | \ell_1\ \beta_1\ u_2\ v_2\ g\ (\beta_{5v} - \beta_1) |$

binary ternary gear_pair fixed_angle



i = R/r

$$\Delta\varepsilon_0 = \begin{vmatrix} \Delta\ell_1 \\ \Delta\beta_1 \\ \Delta u_2 \\ \Delta v_2 \\ \Delta g \\ \Delta(\beta_{5v} - \beta_1) \end{vmatrix} = \begin{vmatrix} c_1 & s_1 & 0 & 0 & 0 & 0 \\ -\frac{s_1}{\ell_1} & \frac{c_1}{\ell_1} & 0 & 0 & 0 & 0 \\ -c_2 & -s_2 & c_2 & 0 & v_2 & 0 \\ s_2 & -c_2 & -s_2 & 0 & -u_2 & 0 \\ 0 & 0 & 0 & +i & (1-i) & -1 \\ \frac{s_1}{\ell_1} & -\frac{c_1}{\ell_1} & 0 & 0 & 1 & 0 \end{vmatrix} \times \begin{vmatrix} \Delta x_A \\ \Delta y_A \\ \Delta x_B \\ \Delta\beta_4 \\ \Delta\beta_5 \\ \Delta\beta_2 \end{vmatrix} = \begin{vmatrix} 0 \\ \Delta\beta_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

Fig. 5.4.11 Modelling the pair of gears in a step-dwell mechanism

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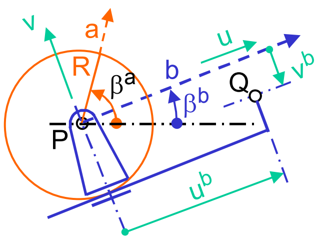
co-ordinates $x^{kT} = | x_P\ y_P\ \beta^b\ x_Q\ y_Q\ i\ \beta^a |$

form parameter $\varepsilon = g$

and u, v see ternary elem.(1)

cont. equation $g = i.(\beta^a - \beta^b) - u^b$



Usually the i-co-ordinate is fixed

1st order continuity equation

$$\Delta\varepsilon = \Delta g = | c^b\ \ s^b\ \ (-v^b - i)\ \ -c^b\ \ -s^b\ \ (\beta^a - \beta^b)\ \ i | \times \begin{vmatrix} \Delta x_P \\ \Delta y_P \\ \Delta\beta^b \\ \Delta x_Q \\ \Delta y_Q \\ \Delta i \\ \Delta\beta^a \end{vmatrix}$$

$$i = \frac{du^b}{d(\beta^a - \beta^b)} = \pm R$$

Fig. 5.4.12 Pinion-and-rack as an element (1), using the transmission coefficient i as a co-ordinate

A variant pair of gears can be defined with the gear radii R_P and R_Q as constants in the equation (5.36) instead of i. The radii must both be understood as co-ordinates then. The major difference is that the gear radii don't have to match with the bar length. The angular relationship of the gear pair remains correct, which can be used to model a chain or belt without slip. Of course the gear radii will normally be used as fixed co-ordinates. By choosing positive or negative values for the radii all types of contact (inner or outer) can be specified. This element will be named further ***Gear-2 element***.

A physical interpretation of the form parameter concerning the gear-relation (5.36) can be made as follows. Consider the special situation that element a has the same direction as element b, see fig. 5.4.10. In this position the form parameter has the same value as the angular position of element c relative to element b. This value appears to be thus an assembly angle and will be noted further as g (gear parameter). A change (deformation) of g can be interpreted as a change of this assembly angle.

An example of a mechanism having a pair of gears is depicted in fig. 5.4.11. Point B lies precisely on the rolling circle of the planetary gear (element 2) and is also guided on a horizontal line through A_0. When the elements 1 (the connecting rod) and 2 are inline, the velocity of point B is zero. With point B in the most right position this is at the same time the contact point of the gears. The sun gear centred in A_0 (element 4) has then a dwell, this gear performs a step-dwell motion. Choosing element types it must be realised that the elements 1, 2 and 4 must have an angular co-ordinate. The sun gear (4) is here a half beam element. The planetary gear a ternary-1 element. The connecting rod is a binary element, but with a half beam connected by a fixed angle. This example demonstrates the various options of modelling, the user may try to find his own favourite model.

5.4.6 Pinion-and-rack

The ratio of the following geometric quantities is constant:
- the rotation of the pinion (gear, element a) relative to the rack (element b), and
- the distance covered by the contact point along the rack .

Analogous to the pair of gears a relation like

$$i = \frac{\Delta u^b}{\Delta\beta^a - \Delta\beta^b} = \pm R \tag{5.37}$$

could serve as a basis to derive a continuity equation. The "gear parameter" g, which should be considered as a prescribed (undeformed, constant) quantity, is defined then by:

$$\begin{aligned} \Delta g &= i \cdot (\Delta\beta^a - \Delta\beta^b) - \Delta u^b = 0 \\ g &= i \cdot (\beta^a - \beta^b) - u^b \end{aligned} \tag{5.38}$$

There are various modelling options for the angles β and the distance u.
First it will be assumed that both angles β^a and β^b are co-ordinates, which are already present in the elements a and b. Typically a is a half beam element and b a ternary-1 element. To create and to calculate the (extra) continuity equations of this composite Rack-1 element the quantity i must be taken in the co-ordinate vector, see fig. 5.4.12. The first order continuity equations can be derived now using the equation for u of the ternary-1 element. The "third point" Q (called point R in fig. 5.4.5) will normally be chosen to lie on the rolling line of the rack. In that case the value of v must be prescribed to have the same value as (–)R. There is however no objection to choose another value for v. Note that the sign of R determines at which side of the pinion the contact line lies.

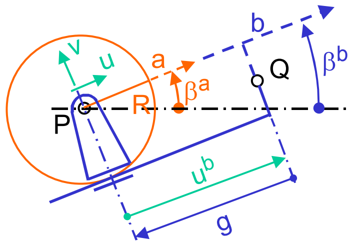
Aid to memory: Consider the pinion and rack separated from the rest of the mechanism. When a positive (relative) rotation corresponds to a longer distance, the value i is positive, otherwise negative.

$g = i.(\beta^a - \beta^b) - u^b$

In case that $\beta^a = \beta^b \rightarrow$

$g = -u^b$

In this position g is the (negative) assembly length of the rack



g remains constant during motion

Fig. 5.4.13 Interpretation of the form parameter g as the initial rack length

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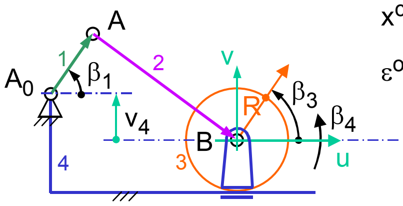

$x^T = | x_B \; y_B \; \beta_4 \; x_{Ao} \; y_{Ao} \; i \; \beta_3 \; x_A \; y_A |$

$x^{cT} = | x_B \; y_B \; \beta_3 \; x_A \; y_A |$

$\varepsilon^{oT} = | \ell_1 \; \beta_1 \; \ell_2 \; v_4 \; g |$

binary (ℓ_1, β_1), binary (ℓ_2), ternary (v_4), pinion-and-rack (1) (g)

$i = -R$

$$\Delta\varepsilon_0 = \begin{vmatrix} \Delta\ell_1 \\ \Delta\beta_1 \\ \Delta\ell_2 \\ \Delta v_4 \\ \Delta g \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & c_1 & s_1 \\ 0 & 0 & 0 & -\frac{s_1}{\ell_1} & \frac{c_1}{\ell_1} \\ c_2 & s_2 & 0 & -c_2 & -s_2 \\ s_4 & -c_4 & 0 & 0 & 0 \\ c_4 & s_4 & -R & 0 & 0 \end{vmatrix} \times \begin{vmatrix} \Delta x_B \\ \Delta y_B \\ \Delta\beta_3 \\ \Delta x_A \\ \Delta y_A \end{vmatrix} = \begin{vmatrix} 0 \\ \Delta\beta_1 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

Fig. 5.4.14 Modelling the pinion-and-rack in a geared slider-crank mechanism

A physical interpretation of g can be found when the orientation of the pinion is assumed to be along the rack ($\beta^a = \beta^b$). In that situation the length u^b can be recognised as the (negative) assembly length, see fig. 5.4.13.

A second way of modelling the pinion and rack is based on the Ternary-2 element. In this case the angle β^b is not available as a co-ordinate, but as a form parameters. The equation (5.38) can still be composed, but now with the help of composition rule (5.33). This composite element will be called further Rack-2 element.

The third modelling option is that the composition of eq.(5.38) is done by the user, based on elements that have the required angles (as a form parameter) and the length available. The constant i can be taken into account as multiplication factor for the single form parameters.

In fig. 5.4.14 an example of a mechanism is presented in which the pinion-and-rack has been modelled with the Rack-1 composite element. Elements 1 and 2 can simply be binary elements. Point B slides horizontal due to the horizontal direction of element 4 (the ternary-1 element).

5.4.7 Belt element

In paragraph 5.4.5 a remark has been made on the Gear-2 element, which may serve to model a belt. But a specific definition of a belt element is certainly possible, see fig. 5.4.15. The form parameter ℓ^{**}, the belt length (including partial circumference of the wheels) can be prescribed. Derivation of the continuity equation for ℓ^{**} will be omitted here, but it can be noticed that

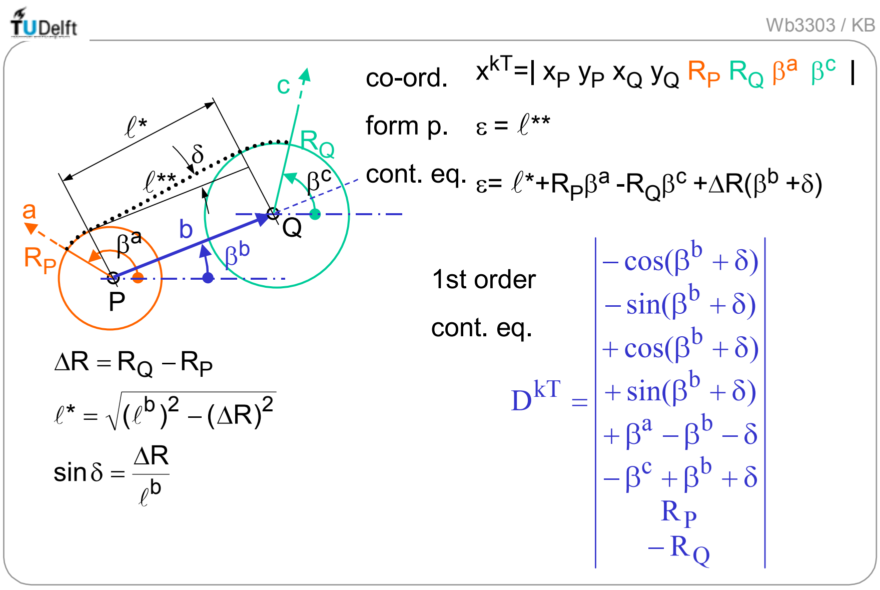

Fig. 5.4.15 Belt element

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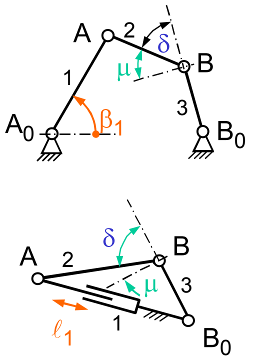

μ =pressure angle

δ =transmission angle

$D^c =$

	x_A	y_A	x_B	y_B	
	c_1	s_1	0	0	ℓ_1
	$-s_1/\ell_1$	c_1/ℓ_1	0	0	β_1
	$-c_2$	$-s_2$	c_2	s_2	ℓ_2
	0	0	c_3	s_3	ℓ_3

$$\mathrm{Det}D^c = \frac{\sin(\beta_3 - \beta_2)}{\ell_1} = \frac{\sin\delta}{\ell_1}$$

Fig. 5.5.1 Determinant of matrix D^c
(Four-bar mechanism and slider-crank mechanism)

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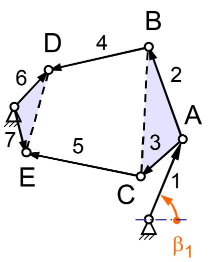


x_A	y_A	x_B	y_B	x_C	y_C	x_D	y_D	x_E	y_E	
c_1	s_1	0	0	0	0	0	0	0	0	ℓ_1
$-\frac{s_1}{\ell_1}$	$\frac{c_1}{\ell_1}$	0	0	0	0	0	0	0	0	β_1
$-c_2$	$-s_2$	c_2	s_2	0	0	0	0	0	0	ℓ_2
$-c_3$	$-s_3$	0	0	c_3	s_3	0	0	0	0	ℓ_3
$\frac{s_2}{\ell_2}-\frac{s_3}{\ell_3}$	$\frac{c_3}{\ell_3}-\frac{c_2}{\ell_2}$	$-\frac{s_2}{\ell_2}$	$\frac{c_2}{\ell_2}$	$\frac{s_3}{\ell_3}$	$-\frac{c_3}{\ell_3}$	0	0	0	0	$\beta_2 - \beta_3$
0	0	$-c_4$	$-s_4$	0	0	c_4	s_4	0	0	ℓ_4
0	0	0	0	$-c_5$	$-s_5$	0	0	c_5	s_5	ℓ_5
0	0	0	0	0	0	c_6	s_6	0	0	ℓ_6
0	0	0	0	0	0	$-\frac{s_6}{\ell_6}$	$\frac{c_6}{\ell_6}$	$\frac{s_7}{\ell_7}$	$-\frac{c_7}{\ell_7}$	$\beta_6 - \beta_7$
0	0	0	0	0	0	0	0	c_7	s_7	ℓ_7

$$\mathrm{Det}D^c = \frac{\sin(\beta_4 - \beta_6)\cdot\sin(\beta_3 - \beta_5)}{\ell_1\ell_2\ell_7} - \frac{\sin(\beta_5 - \beta_7)\cdot\sin(\beta_2 - \beta_4)}{\ell_1\ell_3\ell_6}$$

Fig. 5.5.2 Determinant of matrix D^c, Stephenson-2 mechanism

- Angle β^b of the connecting rod b is based on the binary element. This angle is thus available as a form parameter (function of the co-ordinates of points P and Q). In the derivatives of the continuity equation the chain rule has been used.
- The value ℓ^* (belt length between the contact points on the wheels) must be calculable. Due to possibly a negative square root not all values for R_P, R_Q and ℓ^b are allowed. A zero value for ℓ^b will fail anyhow, but this value will not be acceptable too for a binary element. The calculation problems reflect the physical impossibilities of this element, but the user remains responsible for correct values.
- Length and angle of the connecting rod b can be understood either as prescribed or as dependent form parameters. With variable length this element acts as a rack, but here with two pinions.

Further appreciation of this element can be found in the dynamic properties. It can be remarked here that this belt can both push and pull. Consequences for (internal) forces will be discussed in chapter 8 (dynamics).

5.5 Limit positions, movability and transfer quality

5.5.1 General Theory

The iterative computation scheme of figure 5.3.6 shows clearly that only one mathematical operation can be responsible for the movability problem: the matrix inversion of matrix D^c. The following conclusion can be drawn:

> *If an input variable of a mechanism has reached its limit position, then the determinant of matrix D^c is zero.*

The value of this determinant can serve also as a measure for (kinematic) transfer quality, comparable with the pressure angle μ in the four-bar linkage, see figure 5.5.1. (The pressure angle is in general the acute angle between the direction of motion and the direction of the driving force, as mentioned earlier in chapter 2.4). It will be clear that μ should preferably be small, μ =90° indicates a limit position for the crank (angle β_1).

The calculation of the determinant can of course be done numerically by computer program. An algebraic expression for the determinant provides usually better insight. Such an algebraic expression can be found easier with the help of a decomposition of the matrix. The decompositions can be used when a non-diagonal block with zeroes can be recognised in the matrix.

$$D^c = \begin{vmatrix} P & 0 \\ Q & R \end{vmatrix} = \begin{vmatrix} P & 0 \\ 0 & I \end{vmatrix} \times \begin{vmatrix} I & 0 \\ Q & R \end{vmatrix} \text{ and thus is}$$

$$\det D^c = \det P \times \det R \tag{5.39}$$

The diagonal blocks P and R must be square. The square block I is the unity matrix (diagonal coefficients are 1, other coefficients are zero).

The determinant of the D^c matrix of a four-bar linkage as depicted in figure 5.5.1 can then be written then as

$$\det D^c = \frac{\sin(\beta_3 - \beta_2)}{\ell_1} = \frac{\sin\delta}{\ell_1} = \frac{\cos\mu}{\ell_1}$$

This equation provides precisely the same information as the pressure angle: when the bars 2 and 3 are in-line then crank 1 is in a limit position.

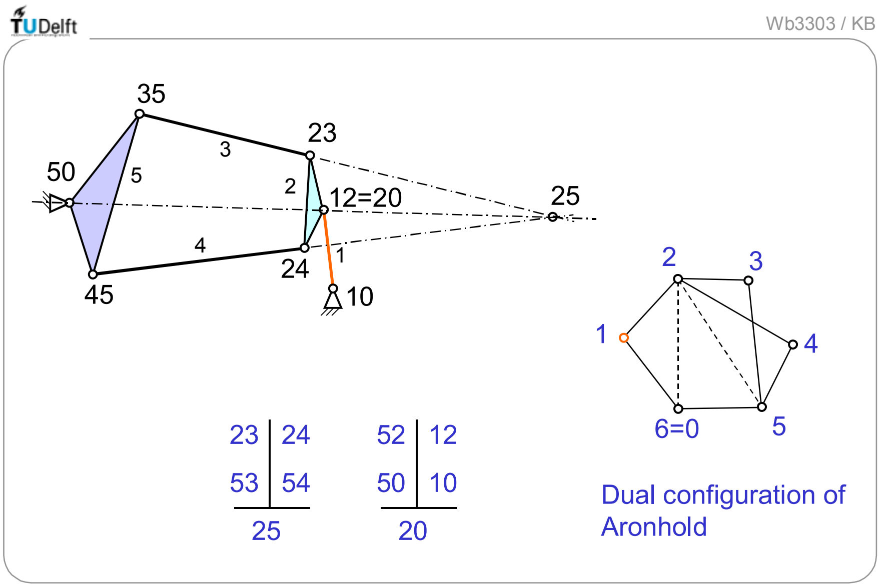
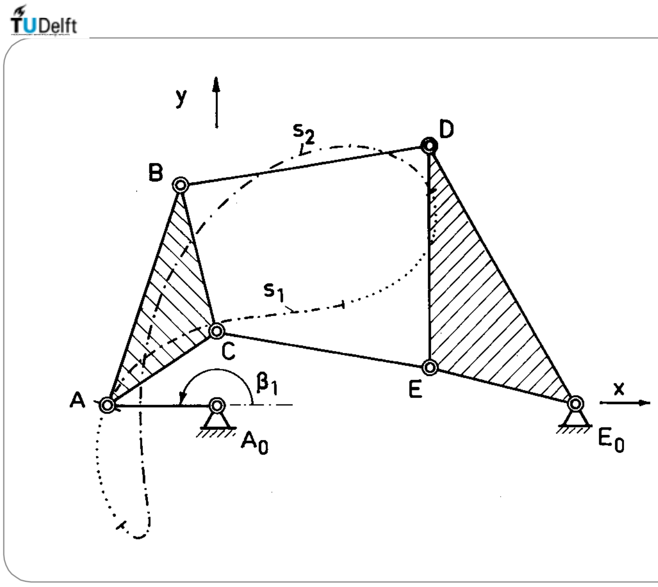


Fig. 5.5.3 Limit position for the Stephenson-2 mechanism (relative poles 12 and 20 coincide)



Initial geometry

	x	y
A_0	0	0
A	-3	0
B	-1	6
C	0	-2
D	6	7
E_0	10	0
E	6	1

Fig.5.5.4 Case study: limit positions of a Stephenson-2 mechanism

Suppose now one of the other prescribed form parameters is applied for input motion, for instance ℓ_1. Of course the angle β_1 is then a fixed angle, so the slider-rocker mechanism of fig. 5.5.1 (lower part) corresponds to this description. This mechanism has the same matrix D^c and thus the same determinant expression.
One could argue which quantity, μ or $\cos\mu$, is a better measure to express the (kinematic) transfer quality. But mind that the determinant has also the factor $1/\ell_1$, which is not constant in the slider-rocker mechanism.
Even more important is the fact that such scale factors are the reason that the value of the determinant may not be used as an absolute measure for transfer quality. Inspection of the behaviour however will certainly provide the positions where the determinant is (almost) zero and thus where the limit positions are. The transfer quality in these positions can be regarded as “relatively poor”.

5.5.2 Case study: the Stephenson-2 mechanism

In literature it is well known that this mechanism shows unexpected limit positions. This is a good reason to verify the results with the determinant approach. In [5.1] such a limit position has been explained with the help of the relative poles, see fig. 5.5.3. As a result it could be formulated:

> *When the point 12 (end-point of the driving crank) lies on the connection line of the poles 50-25, then the crank is in a limit position. In that situation the instantaneous centre of rotation of link 2 coincides with point 12.*

The crank connects then two fixed points and is therefore immovable. This can be verified easily with the dual configuration of Aronhold (see chapter 4.2).
To find a comparable result with the determinant evaluation the mechanism has been modelled according figure 5.5.2. The decomposition rule (5.39) does help, but an 8x8 matrix without a block of zero’s remains, and this matrix part had to be evaluated manually. Finally the following algebraic equation has been found:

$$\mathrm{Det}D^c = \frac{\sin(\beta_4-\beta_6)\cdot\sin(\beta_3-\beta_5)}{\ell_1\ell_2\ell_7} - \frac{\sin(\beta_5-\beta_7)\cdot\sin(\beta_2-\beta_4)}{\ell_1\ell_3\ell_6} \tag{5.40}$$

Obviously this result is correct. When (5.40) is calculated in a situation like in figure 5.5.3, the answer is zero [5.2]. When the sub-chain 2-4-6 (fig. 5.5.2) is in-line, then both terms in (5.40) are zero. The same is true when the sub-chain 3-5-7 is in-line. But it will certainly be possible that both terms are equal, resulting also in $\mathrm{det}D^c = 0$. Typically this is the case in fig. 5.5.3. So far it could however not be proven that equation (5.40) can directly be derived from figure 5.5.3.

For demonstration purpose a special Stephenson-2 mechanism has been drawn in figure 5.5.4. Relative to plane E_0ED point A describes a four-bar coupler curve, which has been drawn in the figure. The two-link subchain E_0A_0A prevents that point A can follow the whole coupler curve. Only the branches s_1 and s_2 can be reached, and it depends on the way of assembling which one will be passed. Intentionally the point E_0 was chosen such that it lies (almost) on the evolute of the coupler curve (concerning branch s_2). This provides (almost) a geometry conflict and thus an almost zero determinant. The determinant behaviour (5.40) has been calculated for both assembly possibilities, see fig. 5.5.5. The branch s_1 has obviously no transfer problems, while the branch s_2 shows clearly two almost-zero situations for the determinant. The third curve in fig. 5.5.5 results when the fixed link A_0E_0 will be increased by 1.5. Now a full revolution of crank A_0A is impossible: the crank has two limit positions. Note that there are still two almost-zero situations, near 90 and 270 degrees respectively.

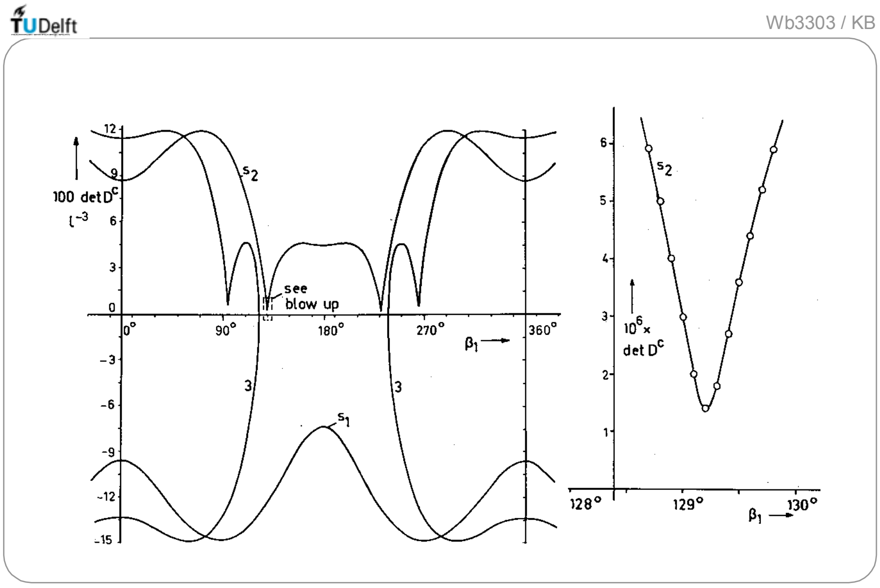


Fig. 5.5.5 Det D^C behaviour of the Stephenson-2 mechanism

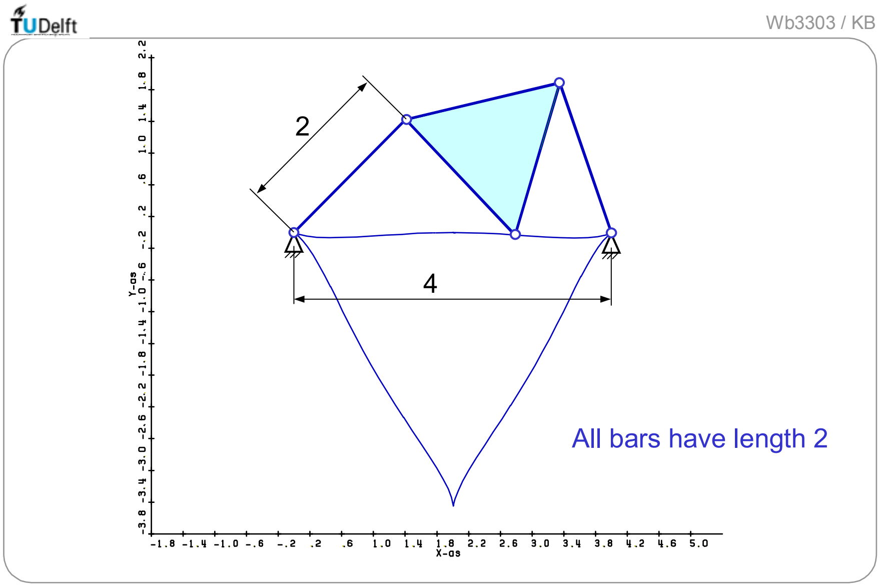


Fig. 5.5.6 Typical non-Grashof mechanism with coupler curve

5.5.3 Position analysis of mechanisms with limit positions

A mechanism, which is already known as a good solution for a design problem, will usually not encounter driving problems during (kinematic) analysis. Such driving problems may typically occur when the type of mechanism is still under discussion or has to be investigated for its properties such as limit positions. A possibility to move the mechanism “just as can be moved” is then very helpful.
A slight modification of the mechanism model can provide this. The idea is that the input motion is not to be determined by a quantity that has been specified by a user, but should be detected before the mechanism will be moved to the new position. This input can be a different one in every new position, so the mechanism keeps going anyhow.

For the detection of a proper input the following procedure could be adopted.

- Remove all regular inputs, that means disregard the continuity equations for prescribed input motion. The matrix D^c has now one or more rows (n, the number of DOF’s) less.
- During inversion of this D^c matrix the columns will be reduced towards a unity matrix, but there will be n columns left to reduce. The co-ordinates involved with these columns are the ones that can be used to add a temporary input.
- Co-ordinates itself are not regarded as input motion in the theory. The temporary inputs should be modelled as simple elements that provide a displacement in the direction of these co-ordinates. An artificial element, consisting of just this co-ordinate x can be helpful. The continuity equation is simply:

  $$\varepsilon = x$$

  and the contribution to the matrix D^c is a row with zeroes, but a 1 at the column as indicated.
- After the new equations have been added, the reduction process of the matrix can be continued (In mathematics this procedure would be called: the kernel of the map $E^o \rightarrow X^c$ has been determined to detect the direction of free motion).
- A new mechanism position can be obtained when stepsizes for the new inputs have been submitted. Such a stepsize should be derived from a total displacement of co-ordinates, to be specified by a user.

In case the mechanism has one DOF, a stepsize Δs can be defined as the total step in the (Euclidean) co-ordinate space:

$$\Delta s = \sqrt{\sum_i (\Delta x_i)^2}$$

This step must be related to a step for the specific input motion $\Delta\varepsilon^m$. Since the corresponding row of the inverted D^c matrix contains the partial derivatives $dx^c/d\varepsilon^m$, the derivative of s is the norm of this row

$$\frac{ds}{d\varepsilon^m} = \sqrt{\sum_i \left(\frac{dx_i^c}{d\varepsilon^m}\right)^2} \quad \text{and}$$

$$\Delta\varepsilon^m = \left(\frac{ds}{d\varepsilon^m}\right)^{-1} \cdot \Delta s$$

is the step to be applied at the temporary input.

TUDelft Wb3303 / KB

TOPOLOGY

ELEM	1	BIN	1	2	7	8
	2	BIN	7	8	9	10
	3	BIN	3	4	9	10
	4	BIN	5	6	9	10

co-ordinate numbers of the elements

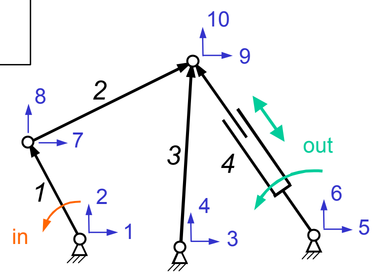


Fig. 5.6.1 Topology table, part 1 (element specification)

TUDelft Wb3303 / KB

FORM	1	LENL	1	1	1.0
	2	ANGB	1	4	1.0
	3	LENL	2	2	1.0
	4	LENL	3	3	1.0
	5	LENL	4	-1	1.0
	6	ANGB	4	-2	1.0
NRDOF	1				

form par. type
element number
destination (row nr. D^c)
multiplication factor

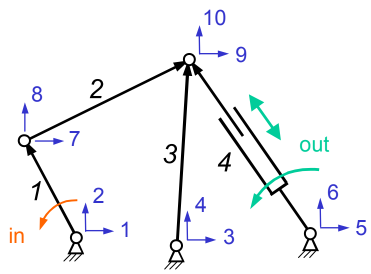


Fig. 5.6.2 Topology table, part 2 (form parameter specification)

The step direction is another problem. The co-ordinates should continue to move in the same direction until they have reached a limit position. This can be guaranteed when the temporary input of the *previous* step will be considered too.

A typical example of a mechanism is depicted in fig. 5.5.6. It is a non-Grashof mechanism, which means that none of the links can make a full revolution. The coupler point traces a closed curve, which can be found by the procedure as described above. Theoretically it can be proved that it is a triangular-like curve, with three-folded symmetry and with three cusps.

5.6 The program RUNMEC

5.6.1 Introduction

The computer program RUNMEC (RUNning MEChanisms) supports the theory as described in this book. The present version of the program reads all information from the input file, which has to be prepared by the user (free format text file) and presents the results on the screen. The input file contains information like:
- tables for the description of the mechanism (topology and geometry),
- commands for the motion of the input links (DOF's) and
- commands for the desired output.

The program can be run in several "modes": either kinematics mode or dynamics mode. In kinematics mode a further distinction is either analysis or optimisation. In dynamics mode either analysis or simulation can be chosen. These switches refer to the corresponding parts of the theory. It depends on the mode which information in the input file is required or allowed.

The construction of the input file is as follows:
- The first line must contain the mode. In this chapter the kinematic analysis mode will be used (the first command line should contain the code: KIN ANA).
- A line with the block header word must precede the various tables.

The general syntax structure of all lines is:

KEYWORD argument1 argument 2 ···

The keywords are as commands: they must be typed precisely. The arguments are usually numbers, but sometimes also a command. Further syntax rules are:
- Only one keyword per line is allowed
- Both uppercase and lowercase characters can be used (names are not case sensitive)
- Numbers can be given in free format (Fortran convention, e.g. –3.6, 4E5, -3.d-2)
- Numbers must be separated by blanks (tabs not allowed)
- After a semicolon the remaining text on that line is comment
- Keywords don't have to be repeated (if numbers are not preceded by a keyword, then the previous keyword will be used)
- Angles must be given in radian, length dimension will be according user interpretation.

The tables concerning the mechanism description are needed in the other modes as well. Therefor they will be explained in the next paragraphs briefly.
Detailed information can be found in part II, the manual.

5.6.2 Topology table

A line with the (block header) word TOPOLOGY must precede the topology table, see the example for the mechanism of fig. 5.6.1. The topology table consists actually of three items:

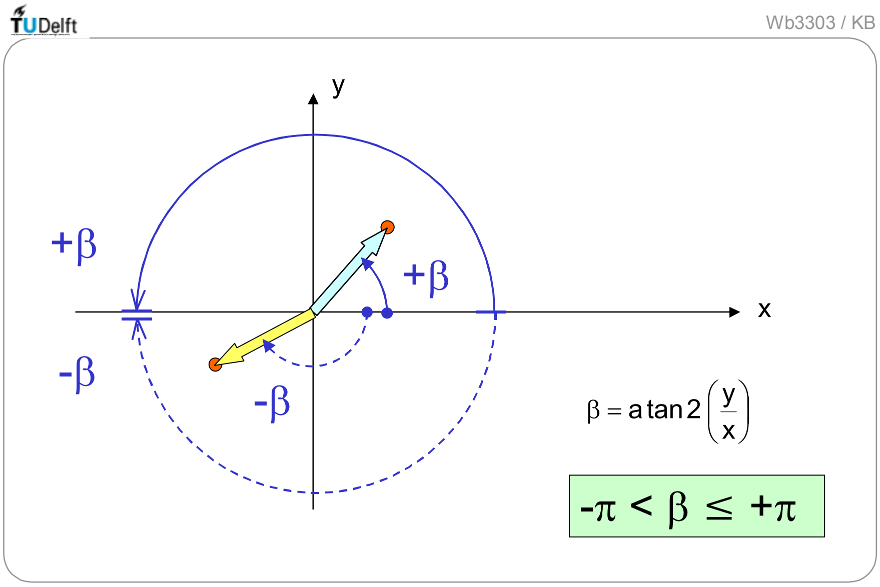


Fig. 5.6.3 Initial value of angular form parameters

- A table for the list of elements (depicted in fig. 5.6.1 also),
  The keyword to define an element is ELEM.
  Elements have three-character names (BIN for a binary element).
  The co-ordinates must be numbered 1,2,3··· in a closed sequence, first the fixed co-ordinates, then the moving co-ordinates. The numbering corresponds precisely to the column numbering of the D-matrix.
- A table with the description of the form parameters (fig. 5.6.2)
  The keyword to take a (single) form parameter into the model is FORM.
  Form parameters have a four-character name (LENL for the length and ANGB for the angle of a binary element).
  The first argument is just a sequence number; the fourth argument specifies the destination (row number of matrix D). Combined form parameters can be created by specifying the same destination for two or more single form parameters (and use the multiplication factors).
  Positive destination numbers must be used to indicate prescribed form parameters.
  Negative numbers are for dependent form parameters (they are optional).
- A specification of the degree of freedom, using the keyword NRDOF.

5.6.3 Geometry table

The header of the block is a line with the word GEOMETRY. The geometry table consists of two items:

- A table with the co-ordinates in the initial position. First the fixed co-ordinates (keyword XFIX), then the moving co-ordinates (keyword XMOV), see the example in table (5.2) in chapter 5.6.6.
- A table (optional) with the values of the prescribed form parameters (keyword PARA). If such a value has been specified, then the moving co-ordinates of the initial position will be adjusted. If the value has been omitted, then the value will be calculated from the co-ordinates (and further kept constant). The initial value of a degree of freedom can be specified here.

Special attention for angular form parameters is required when the PARA command will be used. The angle will be calculated according the arctangent function, with the resulting angle β as close to zero as possible (see fig. 5.6.3):

$$-\pi < \beta \leq +\pi$$

When the user is not aware of this procedure, unexpected results may occur. Suppose for instance that, for convenience expressed in degrees, a certain angular form parameter has been calculated to be –170° and that a value of +190° has been prescribed. The angles +190° and –170° look the same in a figure like 5.6.3. In the calculation procedure however a difference of +190° – (–170°) = 360° will be detected. This difference should be eliminated by the iteration procedure. It is very likely that this is a too big step.
When (angular) form parameters are to be combined, the same problem needs attention. Suppose $\beta_1 = -60°$ and $\beta_2 = +60°$, and that their difference $\beta_1 - \beta_2$ must be prescribed with the PARA keyword. Here a value near -120° must be specified.
Suppose $\beta_1 = 200°$, then this angle must be interpreted as –160°. Now the value of $\beta_1 - \beta_2$ is –220° and not 140°.

5.6.4 Input motion

In kinematics mode a range of steps can be specified for the input motion(s). The initial (adjusted) position gets the number 1. Step 1 displaces the mechanism from position 1 to 2 and so on. The last position gets thus the number n+1.

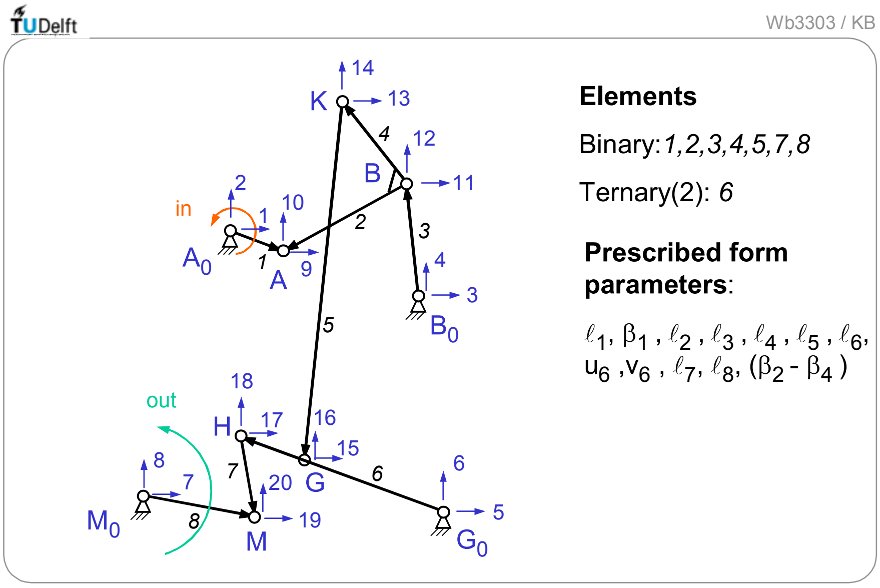


Fig. 5.6.4 RUNMEC-model of the supply mechanism of a punching machine

```
TOPOLOGY
ELEM 1 BIN   1   2   9 10
     2 BIN  11  12   9 10
     3 BIN   3   4  11 12
     4 BIN  11  12  13 14
     5 BIN  13  14  15 16
     6 TR2   5   6  17 18 15 16
     7 BIN  17  18  19 20
     8 BIN   7   8  19 20
FORM 1 LENL   1   1   1.0
     2 LENL   2   2   1.0
     3 LENL   3   3   1.0
     4 LENL   4   4   1.0
     5 ANGB   2   5   1.0 ; form parameter 5 contains (beta2-beta4)
     6 ANGB   4   5  -1.0
     7 LENL   5   6   1.0
     8 LENL   6   7   1.0
     9 LENU   6   8   1.0
    10 LENV   6   9   1.0
    11 LENL   7  10   1.0
    12 LENL   8  11   1.0
    13 ANGB   1  12   1.0 ; input  angle beta1
    14 ANGB   8  -1   1.0 ; output angle beta8
NRDOF 1
```

Table 5.1 Topology table of the supply mechanism

For equidistant steps the keywords MOV_ABSE and MOV_RELE can be used. Non-equidistant steps can be read from a file (keyword MOV_FUNC), which has to be prepared before. Free motion can be specified with the keyword MOV_FREE.
Input motion, as a function of time, can be specified with the keyword TSTEPE (equal time steps). In each position the input velocity and acceleration is then assumed to be constant. The new position of the input will be calculated with a simple extrapolation using the velocity:

$$\Delta\alpha = \dot{\alpha} \cdot \Delta t$$

Non-equidistant time steps can be read from a file, allowing then also to give velocity and acceleration of the input more specifically (keyword MOV_FUNC). In chapter 7.1 more information about the file will be given.
Note: in version 4.0 of Runmec the keywords for moving the mechanism have been modified slightly. The main differences are:

- keyword TSTEPE must be used always as the first keyword in the Movement block, and
- each drive (DOF) requires a following keyword (they can be different).

5.6.5 Output of transfer functions and other calculated results

The iterative procedure to calculate transfer functions requires that results must be stored for output processing afterwards. This can be invoked selectively with output commands in the input file (keywords BUFxxx). A long list of options has been created to be helpful in the kinematic and dynamic mode. The last argument with these keywords is an output buffer number, which can be chosen freely by the user. A buffer number refers then to a function (list of values) of the chosen kind.
The output functions can be given a destination like:

- Printfile (keyword PRINTBUF, works for all buffer numbers).
- Dataset (keyword SAVEDATA, works only for specified buffers).
- Picture on screen. Between the keywords BEGINPIC and ENDPIC a list of functions can be specified, which will be drawn in the same picture. Actually there are two types of pictures
  - ⇒ for periodic functions (example in fig. 5.6.6)
  - ⇒ for two-dimensional curves. In this picture it is also possible to depict the mechanism itself and, in case of a coupler curve, the local hodograph of velocities (first order vectors in kinematics). See the examples of figures 5.5.6 and 5.6.5.

5.6.6 Example: supply mechanism for a high-speed punching machine

The required motion of the output link is an oscillating, periodic rotation $\delta(\alpha)$. A one-directional clutch transforms this motion into a stepping motion of a sprocket wheel. The metal strip to be punched is connected to the belt and will be punched during the dwell (the return stroke of the mechanism). A permanent brake is installed at the sprocket wheel, to follow the deceleration of the oscillating link properly. The case study concentrates on the motion of the mechanism generating the oscillating motion.

The high-speed operation (600 strokes/minute) requires that the output link has a very smooth motion. The transfer of the driving moment between output link and sprocket wheel can only be continuous (no jerk) when the acceleration of the output link $\ddot{\delta}$ is zero in both limit positions. According (1.7), for constant rotation speed $\dot{\alpha}$ of the input shaft, it is required then that the second order transfer function $\delta^{//} = 0$ in both end points.
Such a motion can be achieved with a Stephenson-3 mechanism, see fig. 5.6.4. Coupler point K describes a symmetrical coupler curve with equal curvature in both symmetry points, see fig. 5.6.5. Link KG has the same length as the radius of curvature, so point G (link G_0H) has an instantaneous dwell in both limit positions (remember chapter 4.6.3). The Four-bar chain G_0HMM_0 has been added to provide stroke amplification. Bar M_0M is finally the output link.

```
GEOMETRY
XFIX 1   0.0
     2   0.0
     3   0.55678
     4  -0.9
     5   1.04091
     6  -6.22864
     7   0.17488
     8  -6.72864
XMOV 9   0.2
    10  -0.34
    11   1.0
    12  -0.2
    13   0.3
    14   0.3
    15   0.55
    16  -6.23
    17   0.84
    18  -6.3
    19   0.45
    20  -7.2
PARA 1   0.4
     2   0.8
     3   0.8
     4   0.8 ; for symmetrical coupler curve the lengths 2, 3 and 4
             ; must be equal
     5  -5.17514 ; the angle kappa = beta2-beta4 in the coupler plane
     6   6.78543
     7   0.71771
     8   0.50120
     9   0.0
    10   0.5
    11   0.64952
```

Table 5.2 The geometry table of the supply mechanism

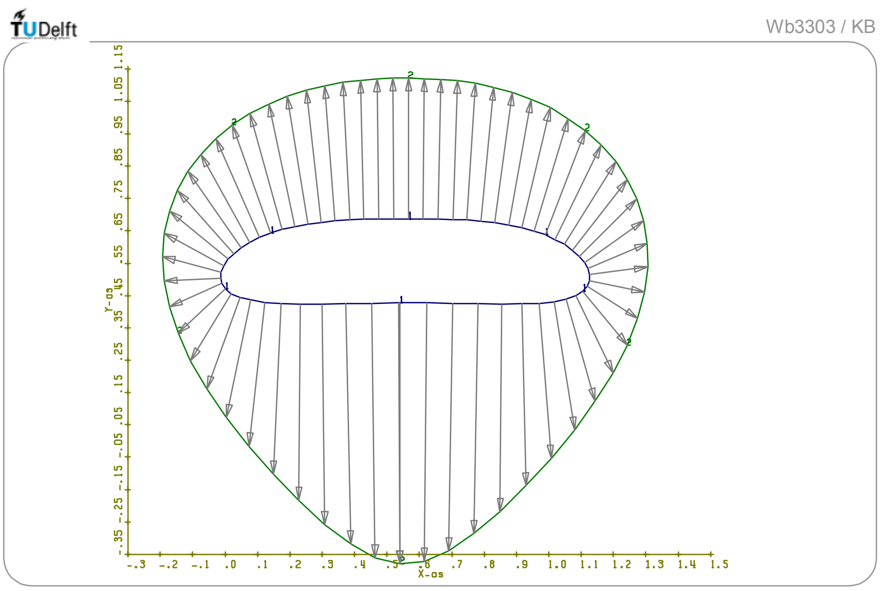


Fig. 5.6.5 Symmetrical coupler curve and hodograph of first order motion (supply mechanism in punching machine)

The design of the four-bar mechanism having the symmetrical coupler curve has been described in [5.4]. The symmetry condition is that the three bars 2, 3 and 4 have equal length (such a mechanism is also known as a λ-mechanism). The angle $\beta_4 - \beta_2$ has been calculated such that the equal curvature exists.

Kinematic analysis is useful to verify the result, for instance the precise behaviour of the output link can be calculated. A RUNMEC model can be made for this purpose, see fig. 5.6.4. The tables for topology, geometry and output commands have been written out, see tables 5.1-5.3. The output of the pictures, as provided by the program, has been depicted in figures 5.6.5 and 5.6.6. It can be noticed that both end points have indeed a dwell, but one dwell is much longer than the other one.

```
STORAGE
 BUFINP  1  99
 BUFE0  -1  100
 BUFKE1 -1  1  101    ; output angle, first order transfer function
 BUFKE2 -1  1  1  102 ;     ,,       , second order    ,,
 PRINTBUF
 ANIMATE film
BEGINPIC
 PLOTHODO  13  14
ENDPIC
BEGINPIC
 PLOTFUNC 99 100
 PLOTFUNC 99 101
 PLOTFUNC 99 102
ENDPIC
```

Table 5.3 The output commands table of the supply mechanism

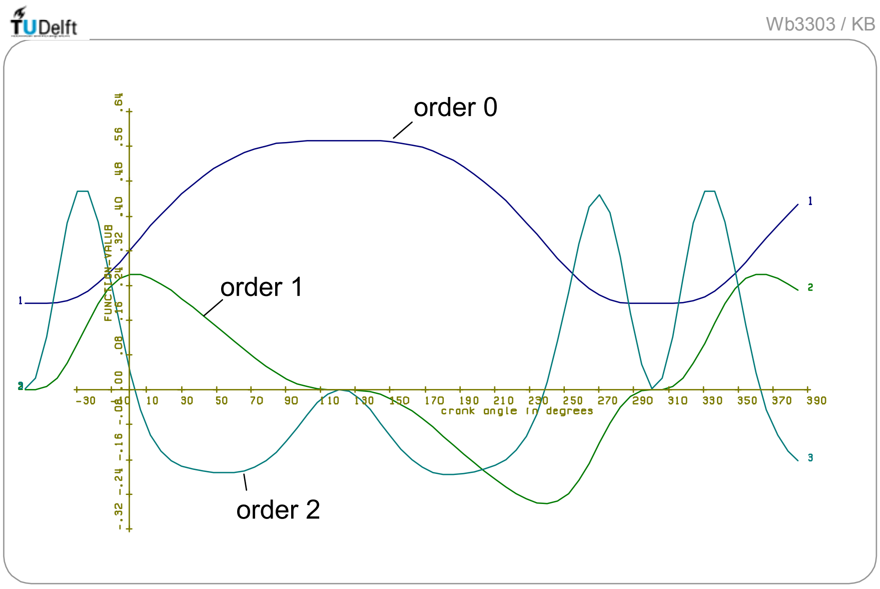


Fig. 5.6.6 Transfer functions of supply mechanism

5.7 References

[5.1]Rössner, W.	Totlagenkonstruktion für sechsgliedrige Zweistandsgetriebe. Maschinenbautechnik 6, S. 105-111 (1957)
[5.2]Klein Breteler, A.J.	Umlauffähigkeit und Übertragungsqualität bei mehgliedrigen Mechanismen. VDI-Berichte Nr.434, 1982.
[5.3]Klein Breteler, A.J.	Movability and transfer quality of arbitrary mechanisms, an application of the FEM-theory. Proceedings of the 6th world congress on Theory of Machines and Mechanisms, New Delhi, 1983.
[5.4]Hagedorn, L.	Vorschubantrieb mit Gelenkgetrieben. Leistungssteigerung bei Schnellstanzautomaten. VDI-Nachrichten 32 (1978). Nr.36, S.10.
[5.5]VDI	Richtlinie 2728 – Lösung von Bewegungsaufgaben mit symmetrischen Koppelkurven. Blatt 1: Übertragungsaufgaben Düsseldorf, 1991.
[5.6]VDI	Richtlinie 2728 – Lösung von Bewegungsaufgaben mit symmetrischen Koppelkurven. Blatt 2: Führungsaufgaben. Düsseldorf, 1991.