4 Special laws and theorems in kinematics 4.1
4.1 Relative poles 4.1
4.2 Dual configuration of Aronhold and Kennedy 4.3
4.3 Theorem of Freudenstein and Meyer zur Capellen 4.5
4.4 Curvature in the Equation of Euler and Savary 4.9
4.5 Pole tangent according to Bobillier and Aronhold 4.11
4.6 Applications with curvature 4.11
4.6.1 Straight line approximation with cycloids 4.11
4.6.2 Horizontal load displacement with a harbour crane 4.13
4.6.3 Dwell mechanism 4.13
4.7 The Bresse circles 4.15

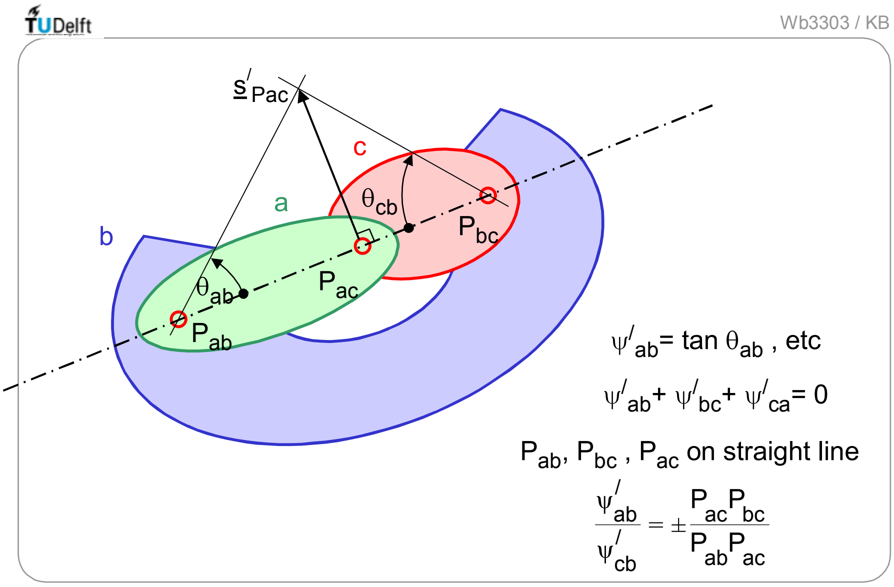

Fig. 4.1.1 Poles, pole distances and first order rotations.
Three planes have three poles on a straight line.

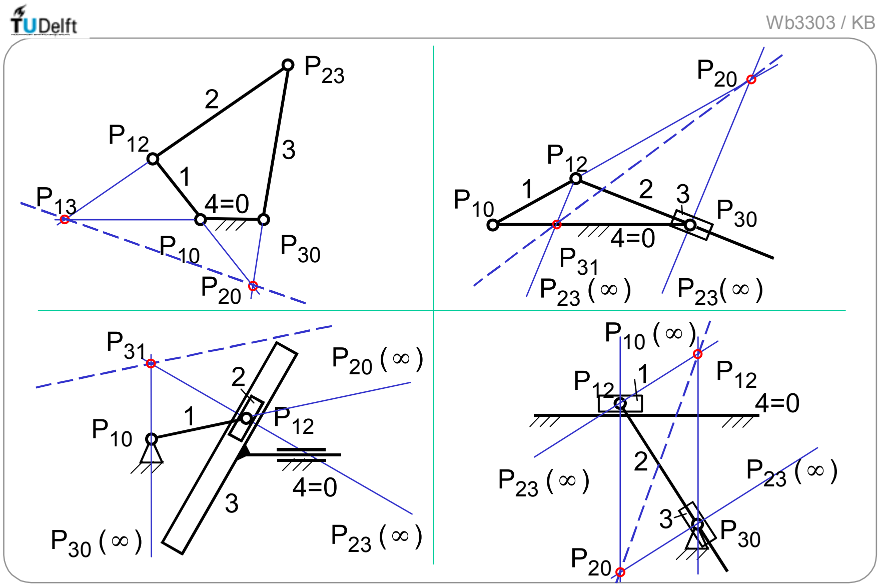

Fig. 4.1.2 First order poles of the Four-bar linkage.
Collineation axis – – – – – – –

4 Special laws and theorems in kinematics

In the past centuries many scientists have contributed to the development of fundamental knowledge of kinematics. In this chapter a selection of laws and theorems will be presented that relate and associate kinematic quantities, and which are therefore beneficial for the design of mechanisms.

4.1 Relative poles

In chapter 3 it was found that a moving plane has always one point without first order displacement (the first order pole). This can be either a permanent pivot point or an instantaneous point. A more general formulation could be that two moving planes always have one point in common without first order motion relative to both planes. Such a point will in general be indicated with *relative pole*.
This idea will be extended now to three moving planes a, b and c, see fig. 4.1.1. The planes can rotate instantaneously (or permanently) around the relative poles P_{ab} , P_{ac} and P_{bc} respectively. The absolute rotations will be indicated with β_a, β_b, β_c and the relative rotations are denoted as

$$\begin{aligned} \psi_{ab} &= \beta_a - \beta_b \\ \psi'_{ab} &= \beta'_a - \beta'_b = \tan\theta_{ab} \end{aligned} \quad \text{etc.} \tag{4.1}$$

Directly from this definition it can be found that:

$$\psi'_{ab} + \psi'_{bc} + \psi'_{ca} = 0 \tag{4.2}$$

A relative pole, like P_{ac} in fig. 4.1.1, has a certain first order motion vector $\underline{s}'_{Pac}$ relative to the third plane b. Since P_{ac} is a common point of both a and c, it will be clear that the two radii to the relative poles P_{ab} and P_{bc} will both be perpendicular to this vector. In other words:

The three relative poles P_{ab}, P_{ac} and P_{bc} are lying on a straight line

Implicitly this kinematic theorem has already been applied to find the first order pole of the coupler plane of a four-bar linkage, see P_{20} in fig. 4.1.2 (upper left). In the same figure some other four-bar types, having sliding parts, and their relative poles have been drawn. The reader may verify that

- All four-bar mechanisms have 6 relative poles.
- A relative pole of two sliding parts lies in infinity.
- A pole of two adjacent links is a permanent pivot. In the four-bar mechanisms the poles P_{20} and P_{13} are non-permanent (instantaneous poles).

The relative pole P_{31} in the four-bar linkage is called sometimes the *help pole*. The line through the poles P_{20} and P_{31} is called the *collineation axis*. This line plays a role in several other constructions in this chapter.

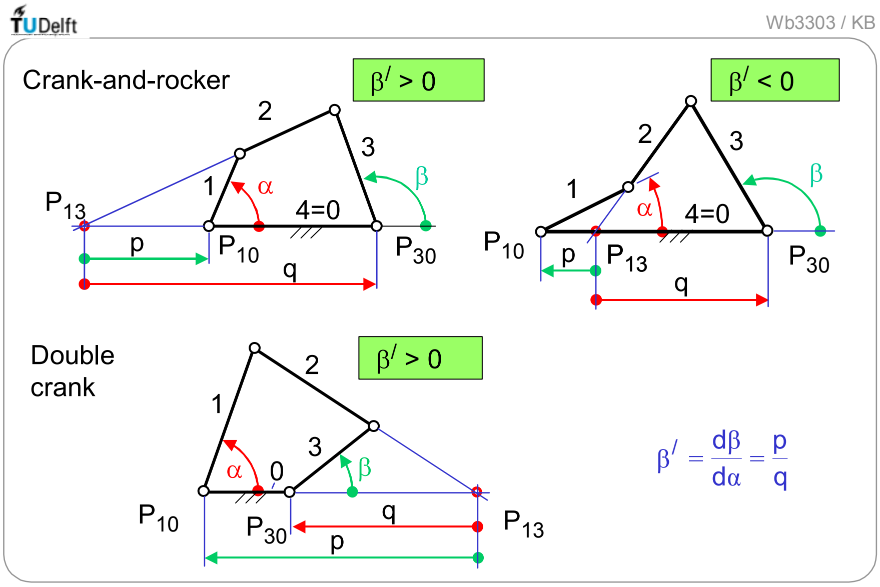

Fig. 4.1.3 First order transfer functions and pole distances p and q

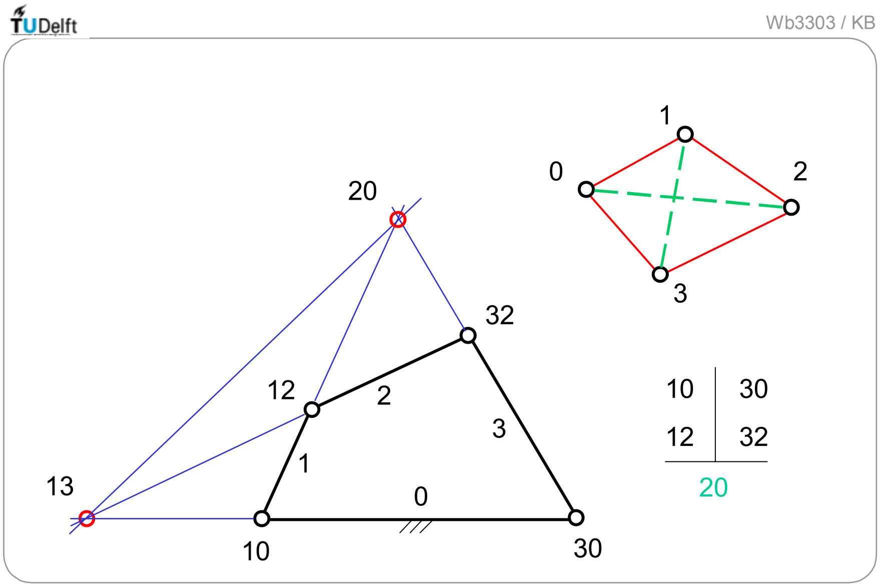

Fig. 4.2.1 Dual configuration of relative poles (acc. Aronhold/Kennedy)

Distances between relative poles are a measure for first order relative rotation between parts, as can be seen easily from fig. 4.1.1. The rotations of a and c relative to b can for instance be expressed as

$$\frac{\psi'_{ab}}{\psi'_{cb}} = \pm \frac{\overline{P_{ac}P_{bc}}}{\overline{P_{ca}P_{ba}}} = \frac{\tan\theta_{ab}}{\tan\theta_{cb}} \qquad (4.3)$$

Since the quantity "distance" is assumed to have no sign, the sign should be derived from the rotation directions θ_{ab} and θ_{cb}.
An application of eq.(4.3) can be found in fig. 4.1.3. Here the transfer function of the four-bar linkage dβ/dα, for this occasion noted as ψ'_{30}/ψ'_{10}, is depicted with pole distances p and q. It can be seen that the position of the help pole P_{13} determines the value of this transfer function. This pole provides therefore a visual help.

4.2 Dual configuration of Aronhold and Kennedy

For simple mechanisms like the four-bar linkage all (relative) poles can be found straightforward. In a more complex mechanism it can be very complicated to find all (relative) poles. Based on the laws of chapter 4.1, Aronhold and Kennedy developed a method to find all poles. They proposed to draw a dual configuration of the mechanism, which includes:

- Draw a link (part) as a point
- Draw a (relative) pole as a connecting line.

The points representing the parts can be arranged at arbitrary places, but the dual figure serves best when all connection lines can be drawn without visual confusion.
The procedure starts with drawing all known connection lines, representing the known poles such as permanent pivots between adjacent links. The dual figure shows then the missing connection lines. Now a visual search can be made to find a quadrilateral with missing diagonals. These diagonals represent two missing poles, which can be found like in a four-bar linkage. Next a new search for a quadrilateral can be made until all missing poles are found.

The method is demonstrated in fig. 4.2.1 for the four-bar linkage. The four bars have been arranged in a dual figure as a quadrilateral with consecutive numbering. The outer edges represent all permanent pivots, which are already known and which can therefore be drawn. The two diagonals represent the unknown poles P_{20} and P_{13}. Their position can each be found by intersecting two lines, specified by the edges of the quadrilateral. For easy application the following method can be used: from point 0 to point 2 in the dual figure you can go via 0-1-2 *and* via 0-3-2. These routes include the poles 01-12 and 03-32, with which the two lines are determined.

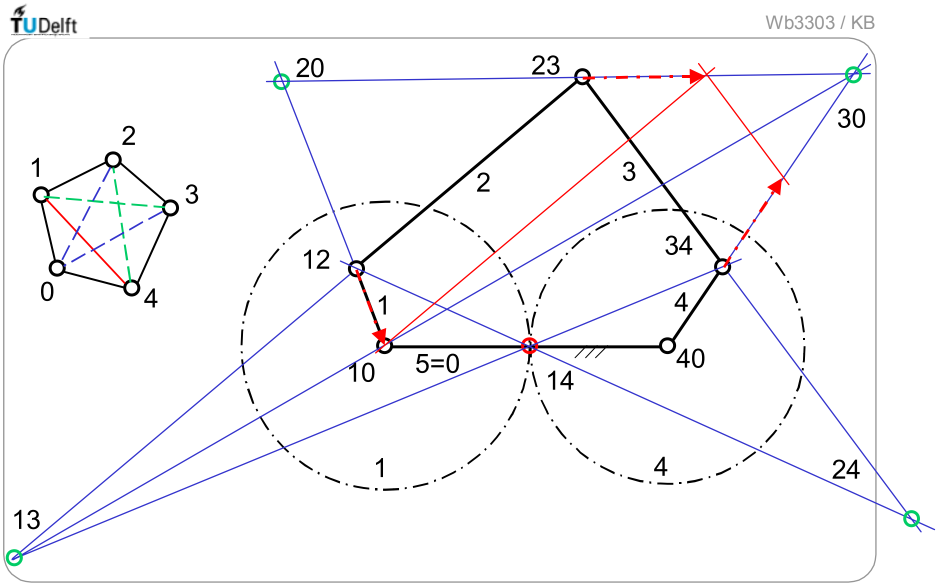

Fig. 4.2.2 Construction of first order poles and motion vectors of a geared five-bar linkage (acc. Aronhold/Kennedy)

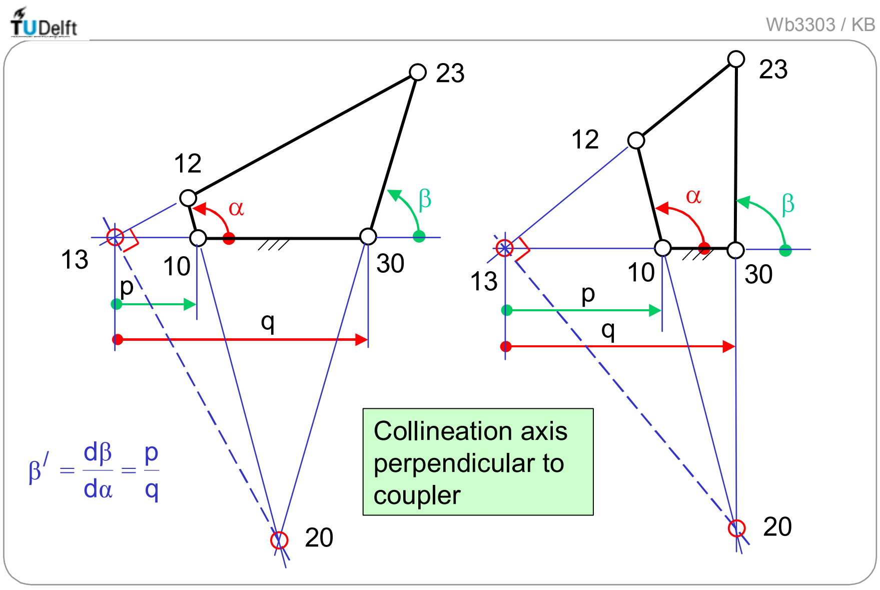

Fig. 4.3.1 Extreme values of first order transfer function dβ/dα (acc. Freudenstein/ Meyer zur Capellen)

A more complex mechanism, a geared five-bar linkage, is presented in fig. 4.2.2. Five points arranged in a polygon represent the five parts. The outer edges all represent permanent pivots, so the outer connection lines can be drawn. The parts 1 and 4 each have a gear, with rolling contact in point 14. So this point 14 is the relative pole between the parts 1 and 4, and the connection line 14 in the dual figure can be drawn too. Assume now that the poles 20 and 30 are to be found. In the dual figure no quadrilateral can be detected with 20 or 30 as a missing diagonal. It is necessary to find other poles first:
Pole 13 can be found (missing diagonal in the quadrilateral 1234): intersection of the lines through 12-23 and 14-34. Pole 24 represents the other diagonal of the quadrilateral, so it can be found by intersecting the lines through 12-14 and 23-24. Now the dual figure contains enough connection lines to find the poles 20 and 30. In figure 4.2.2 pole 30 has been found by intersecting the lines through 13-10 and 34-40. Pole 20 follows from the intersection of the lines through 10-12 and 30-23.
Having found the poles 20 and 30 it is easy now to construct the first order motion of points 23 and 34. Applying the graphical first order construction of Mehmke, just two lines parallel to the bars 2 and 3 are required (in this case link 1 is the driving link).

4.3 Theorem of Freudenstein and Meyer zur Capellen

This theorem deals with the extreme values of the first order transfer function $\beta^/(\alpha)$ of the rocker of a four-bar linkage. In fig. 4.1.3 it became clear that the position of the help pole P_{13} determines the value of $\beta^/$ completely. Obviously $\beta^/$ has an extreme value when the help pole reaches the limits of its region. Freudenstein and Meyer zur Capellen proved that in these positions

> **The first order transfer function $\beta^/(\alpha)$ of the rocker of a four-bar linkage has an extreme value when the coupler and the collineation axis are perpendicular**

Such configurations are depicted in fig. 4.3.1. Note that for a crank-and-rocker mechanism (figure left) the crank point 10 lies inside the region of the help pole. A double crank mechanism (figure right) has a region for the help pole that includes infinity. Now the crank point 10 is outside this region. Consequently the last mechanism has a minimum and a maximum for $\beta^/$ which have both the same sign.

A comparable result can be achieved for the extreme values of the first order transfer function $\gamma^/(\alpha)$ of the coupler (coupler angle γ). Suppose that the coupler and the rocker are completed to form a parallelogram. Consider now the four-bar mechanism with the two added bars instead of the original coupler and rocker. Now the original angles β and γ belong to the coupler and the rocker respectively. In this so-called cognate mechanism the theorem of F/MzC must be true, so we can conclude that

> **The first order transfer function $\gamma^/(\alpha)$ of the coupler of a four-bar linkage has an extreme value when the rocker and the collineation axis are perpendicular**

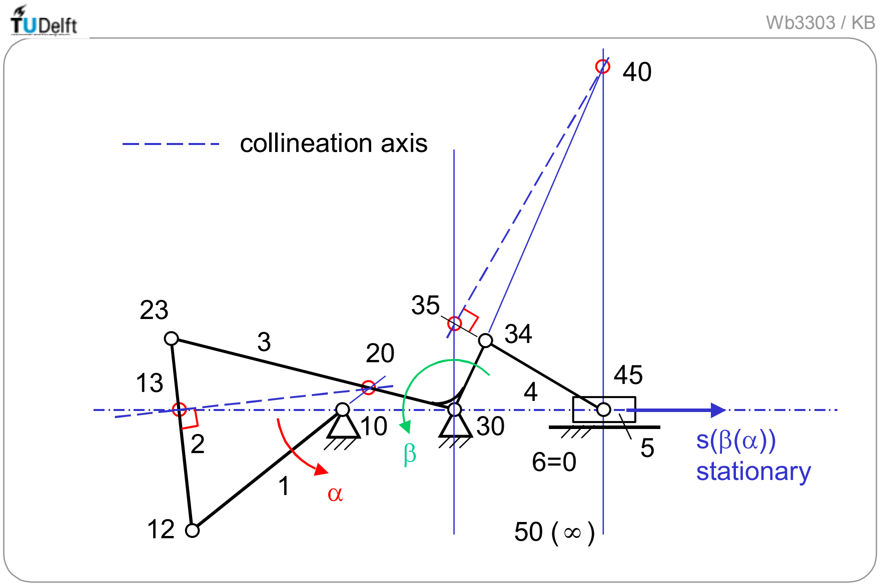

Fig. 4.3.2 *Stationary motion with series connection (simultaneous extreme values of 1st order transfer function)*

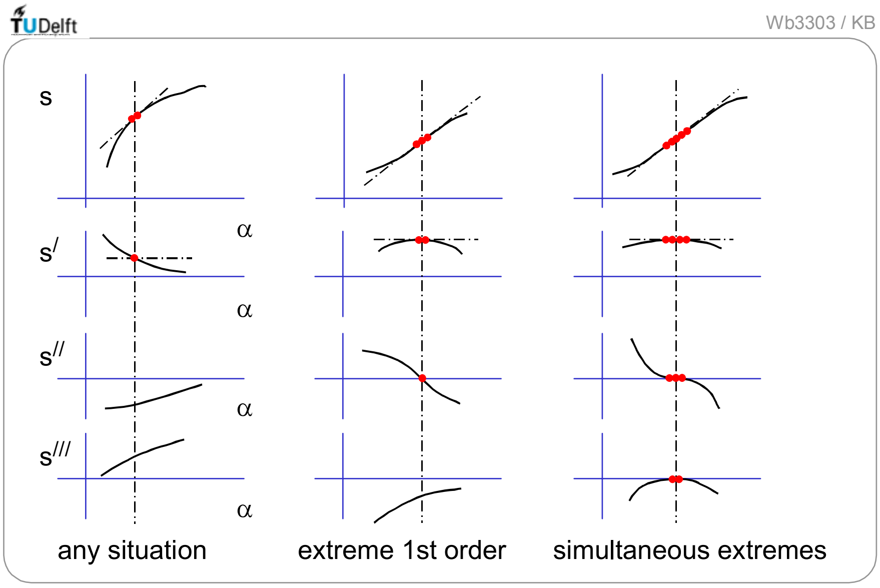

Fig. 4.3.3 *Quality of stationary motion, with number of points* ••• *coinciding with ideal line*

This theorem will be used now to design a mechanism with stationary motion (constant velocity), by putting in series two four-bar linkages which are simultaneously in extreme first order situation. In fig. 4.3.2 a double crank mechanism and a slider-crank mechanism are placed in series. The crank of the slider is driven non-uniformly in such a way that the stationary part of the slider motion will take a longer time. The quality of this stationary motion, which still occurs instantaneously (only in this position) is often measured with "the number of points coinciding with a straight line". This idea is depicted in fig. 4.3.3. In an arbitrary mechanism position a certain first order transfer function does not have an extreme value. There is only one point that coincides with a straight horizontal line (left column in fig. 4.3.3). In the situation considered by Freudenstein and Meyer zur Capellen (column in the middle) the extreme vale in the first order transfer function has at least two coinciding points. With two of these situations in series there could be four coinciding points (column right).
It must be remarked here that:

- To "stretch" the first order extreme behaviour $s^{/}(\beta)$ of the slider it is necessary to combine it with a minimum value in the transfer function of the double crank. Note that this minimum happens when the help pole 13 and pole 30 are then at different side of pole 10.
- Both mechanisms satisfy the condition of Fr/MzC, but are not fully determined by this condition. Some freedom of design still exists, and this can certainly be used to enlarge or optimize the stationary behaviour.

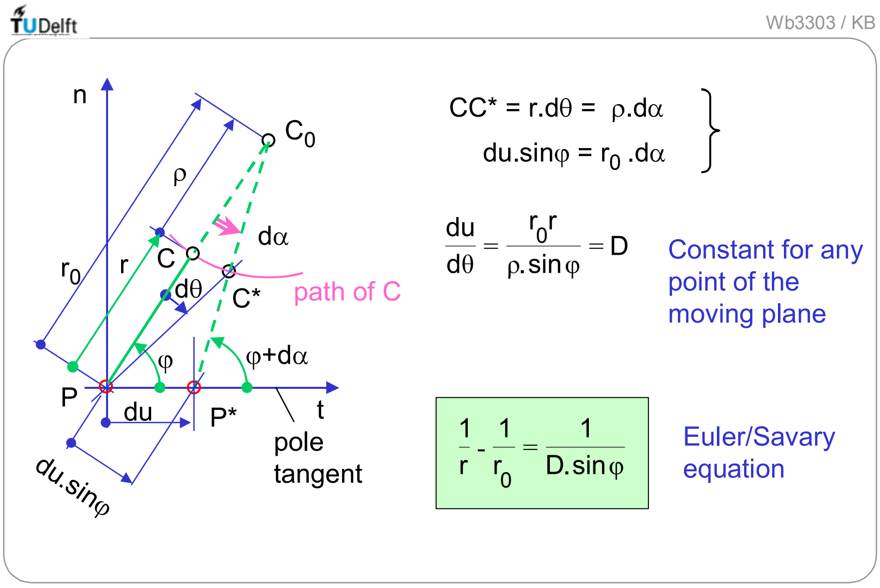

Fig. 4.4.1 *Radius of curvature $\rho = r_0 - r$ acc. Euler/Savary*

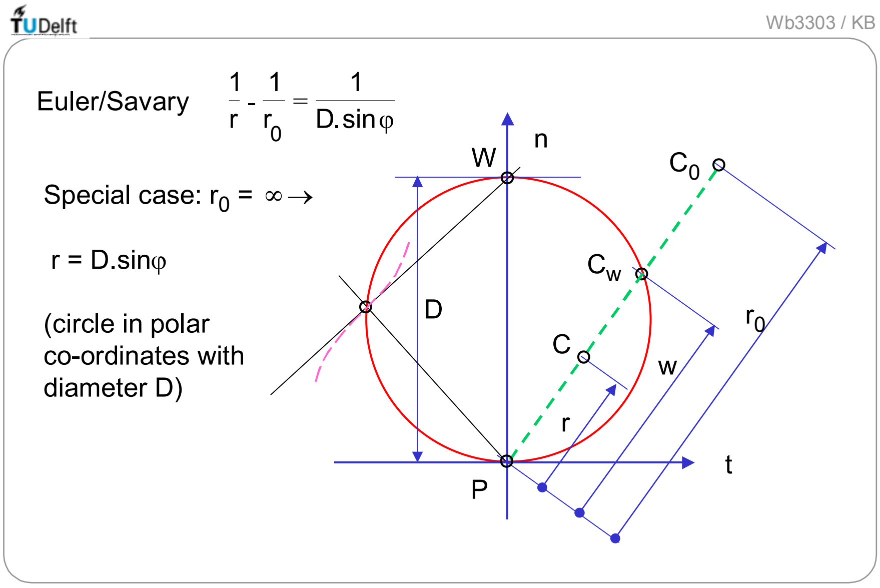

Fig. 4.4.2 *Inflection circle: locus of points of the moving plane with $r_0 = \infty$*

4.4 Curvature in the Equation of Euler and Savary

This equation concerns the radius of curvature of a path, traced by a point that is attached to a moving plane (the radius of the osculating circle, see fig. 3.2.2). The moving point describes the path in a (fixed) co-ordinate system. The origin of the fixed co-ordinate system is chosen in the first order pole, the abscissa in the direction this pole moves (do not confuse with the point of the moving plane coinciding with this pole, since that point has no first order motion). In a general situation a point C of the plane, see fig. 4.4.1, moves perpendicular to radius PC and the centre point C_0 of the curvature is also on radius PC. An infinitesimal displacement of the plane (rotation $d\theta$) lets point C displace to C* (rotation $d\alpha$ around C_0) and P to P* (displacement du along the abscissa). It will be clear that $du/d\theta$ does not depend on the choice of point C, so this is a value of the whole plane in the position to be considered. With the position of points C and C_0 measured in polar co-ordinates (r, φ) and (r_0, φ) respectively, the radius of curvature ρ can be expressed as

$$\rho = r_0 - r \tag{4.4}$$

To avoid the use of absolute values it will be accepted that ρ can have a negative sign. But consequently the values of the radii in this polar co-ordinate system should be allowed then to have a sign. This special version of polar co-ordinate system is indicated with “positive and negative half plane”. Continuing with fig. 4.4.1 it can be found that

$$\frac{du}{d\theta} = \frac{r_0 r}{\rho \cdot \sin\varphi} = D \text{, the characteristic value of the plane, and}$$

$$\frac{1}{r} - \frac{1}{r_0} = \frac{1}{D \cdot \sin\varphi} \tag{4.5}$$

This result is known as the *Equation of Euler/Savary*. Note that the angle φ appears only in the sine function. In this polar co-ordinate system, where the sign is connected to the half plane, it is allowed then to measure it always as an acute, positive angle.

This equation relates the four quantities r, r_0, φ (of a point in the plane) and D (the same for all points). A very nice application is to calculate the radius of curvature ρ of a certain point of the plane. To be able to do so the pole tangent must be determined first (to express the position of the point with r and φ) and the constant D must be calculated. The constant D can indeed be calculated with (4.5), in case that a point with known radius of curvature is given. Usually such a point is available, for instance as the end-point of a crank or a rocker, or as a slider point (r_0 is infinite). How to find the pole tangent in a general situation will be considered further in chapter 4.5.

Now a special property of the moving plane will be treated. Consider the special case that $r_0 = \infty$ (curvature zero), in which eq. (4.5) defaults to

$$r = D.\sin\varphi \tag{4.6}$$

This is the equation, in polar co-ordinates, of a circle through P. The property of “having curvature zero” is thus valid for all points of the moving plane that lie on this circle with diameter D, see fig. 4.4.2. In general such a point changes the sign of the curvature (inflection point). The circle is therefore called *inflection circle*. The point W on the inflection circle opposite to the pole is called the *inflection centre*.

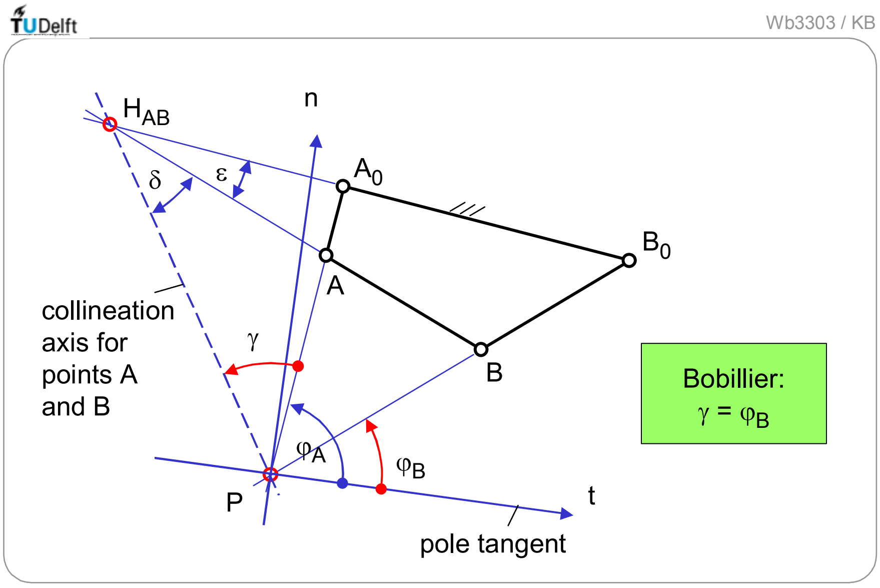

Fig. 4.5.1 *Method of Bobillier/Aronhold for construction of the pole tangent*

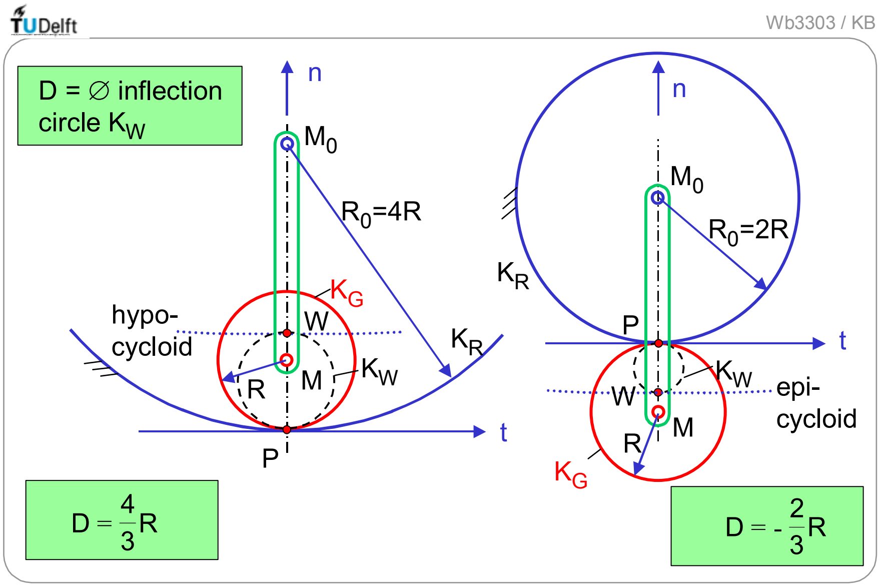

Fig. 4.6.1 *Cycloids approximating a straight line*

4.5 Pole tangent according to Bobillier and Aronhold

A graphical method is proposed to find the pole tangent. The method assumes that two points of the moving plane are given and that their osculating circles are known. These circles implicitly include that the direction of motion of both points is given, so the first order pole P is also known. The coupler plane of a given four-bar linkage typically corresponds to this assumption: the end-points of the crank and the rocker are the points to consider (see fig. 4.5.1). In such a configuration also the help pole H and the collineation axis PH can be drawn. Bobillier and Aronhold considered the two points A en B in the co-ordinate system similar as used for the Euler/Savary equation and found that:

> **The angle between the collineation axis and the radius to the first point is equal to the angle between the pole tangent and the radius to the second point ($\gamma = \varphi_B$ in fig. 4.5.1).**

Applying the notation in polar co-ordinates (r_A, φ_A) for point A and (r_{Ao}, φ_A) for point A_0 the following geometrical relations can be found, see fig. 4.5.1:

$$\frac{\overline{PH}}{r_A} = \frac{\sin(\delta+\gamma)}{\sin\delta} = \cos\gamma + \cot\text{an}\delta \cdot \sin\gamma$$

$$\frac{\overline{PH}}{r_{Ao}} = \frac{\sin(\delta+\varepsilon+\gamma)}{\sin(\delta+\varepsilon)} = \cos\gamma + \cot\text{an}(\delta+\varepsilon)\cdot\sin\gamma$$

With the equation of Euler/Savary it follows

$$\overline{PH} = D\,\sin\varphi_A\,\sin\gamma\,\{\cot\text{an}\delta - \cot\text{an}(\delta+\varepsilon)\} \qquad (4.7)$$

Doing the same for point B and B_0 gives the result

$$\overline{PH} = D\sin\varphi_B\,\sin(\gamma+\varphi_A-\varphi_B)\{\cot\text{an}\delta - \cot\text{an}(\delta+\varepsilon)\} \qquad (4.8)$$

From (4.7) and (4.8) it can be concluded that the two angles γ and φ_B must indeed be equal.

The reader should mind the (positive) direction of these angles as drawn in fig. 4.5.1. The pole tangent and the collineation axis appear to be on opposite sides of the two pole radii PA and PB.

With the construction of Bobillier/Aronhold it becomes possible now, for instance, to calculate the radius of curvature according Euler/Savary. In the next chapter it will be shown that the determination of the curvature can be done pure graphically as well (isn't the construction a graphical equivalent of the formula?).

4.6 Applications with curvature

The theory of Euler/Savary and Bobillier/Aronhold will be applied in the following three examples.

4.6.1 Straight line approximation with cycloids

Cycloids, see figs. 2.7.2 and 2.7.3, offer the possibility to generate symmetrical curves. In such a symmetry point the combination with zero curvature will certainly provide a good approximation of a straight line. Which point of the moving plane (planetary gear) has no curvature is the question of this example.

The example considers two cycloidal mechanisms, see fig. 4.6.1, generating a:

- Hypo-cycloid, gear ratio 4:1 (left part of figure), and a
- Epi-cycloid, gear ratio 2:1 (right part of figure).

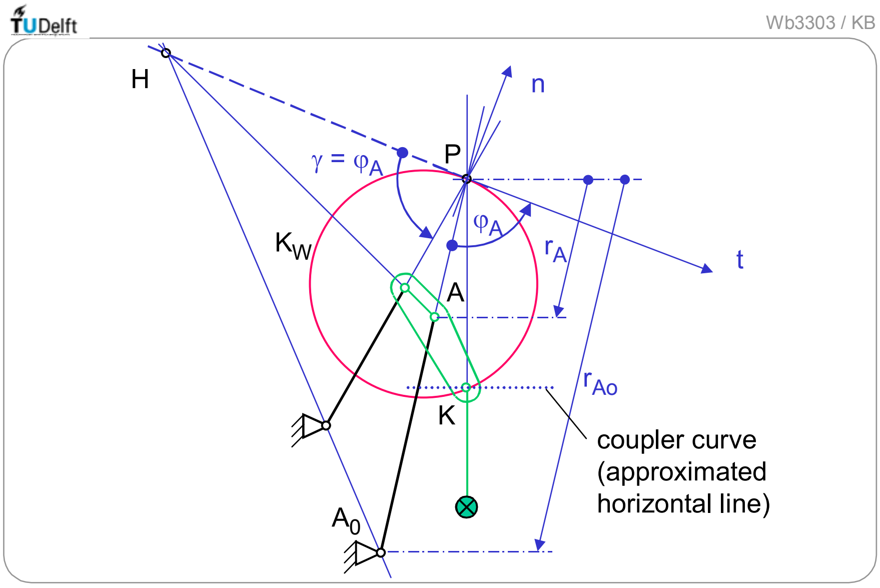

Fig. 4.6.2 The inflection circle used in crane design

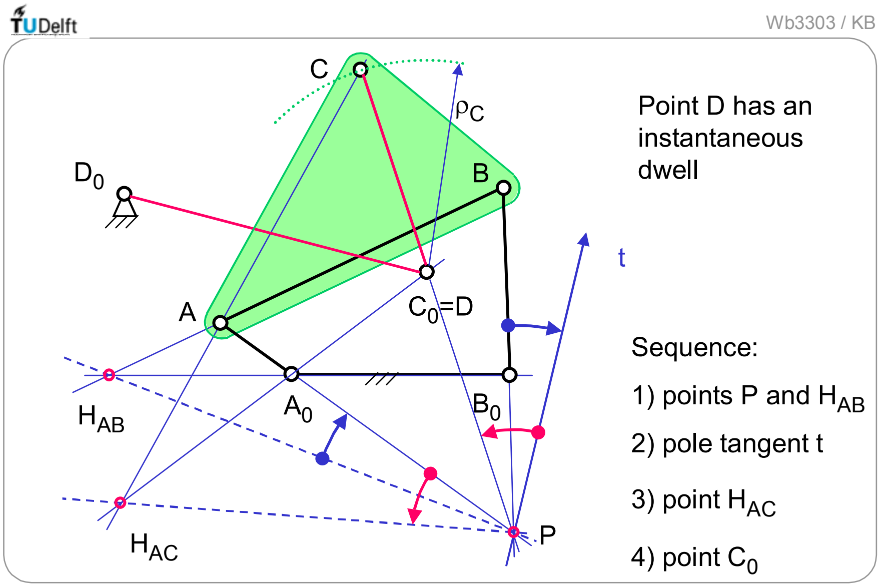

Fig. 4.6.3 Construction of radius of curvature using Bobillier/Aronhold twice (6-bar dwell mechanism)

It can be seen easily that the contact point of the rolling circles (P) is the first order pole of the planetary wheel. Because of symmetry the pole tangent t is horizontal. This provides sufficient information to calculate the diameter of the inflection circle D applying the Equation of Euler/Savary for point M (the centre point of the wheel).
Using the gear radius R to express length it follows then:

r_M = PM = +R (hypo)
-R (epi)
r_{Mo} = PMo= +4R (hypo)
+2R (epi)

In this example the angle φ is 90˚, so sinφ = 1 and eq.(4.5) has the result D = 4R/3 (hypo) and D = -2R/3 (epi) respectively. The inflection centre W is of course the point that combines the two properties of zero curvature and symmetry. Note that the convention of positive/negative sign for the pole radius r is also valid for the diameter of the inflection circle D (in accordance with the idea of the positive and the negative half plane).
The hypo-cycloid of this example approximates a square (with rounded angles).
The epi-cycloid is also known as *Curve of Pascal*. It is a closed, ellipse-like curve with two approximated straight parts at opposite sides.

4.6.2 Horizontal load displacement with a harbour crane

A harbour crane as depicted in fig. 1.2.1 will be considered. It will be assumed that the four-bar linkage is given, see fig. 4.6.2, and that the point K of the coupler plane must be searched that generates the (approximated) horizontal line. In a certain position of the mechanism this point can be determined to lie:

- On a vertical line through the first order pole (this gives the path a horizontal direction), and
- On the inflection circle (this makes the path instantaneously straight)

The graphical construction of the pole tangent according Bobillier/Aronhold is drawn in fig. 4.6.2. The diameter of the inflection circle has been calculated using (4.5) for point A.
(The data used in this equation, as measured on the original drawing, are:
r = 44mm, r_0 = 119mm and φ = 83˚. The result becomes D = 70mm)

Of course the horizontal straight part will only occur in the neighbourhood of the mechanism position as considered. To obtain a practical result the procedure could be applied to a range of mechanism positions. The point K must finally be chosen as an optimal point.

4.6.3 Dwell mechanism

Dwell is the state wherein a point (or a link) has zero velocity for a finite interval of time [1.1]. Instantaneous dwell is the state with *zero velocity and zero acceleration* for an infinitesimal interval of time. This definition excludes a general limit position of a reciprocating motion.

Instantaneous dwell mechanisms can be based on the coupler curve of a mechanism. In any situation it is possible to connect a chain of two extra links between the coupler point and the frame. Choosing one of these links precisely coinciding with the radius of the osculating circle of the coupler point causes that one end-point has an instantaneous dwell. This idea has been depicted in fig. 4.6.3. Point D connecting the two extra links CD and D_0D coincides with the centre of the osculating circle C_0 of point C. In this situation point D, and thereby the whole link D_0D, has an instantaneous dwell.
The problem now is to determine the point C_0. It is for instance possible to calculate the radius of curvature $\rho_C = C_0C$ with the Euler/Savary equation. A pure graphical construction of point C is also possible, applying the construction of Bobillier/Aronhold twice.

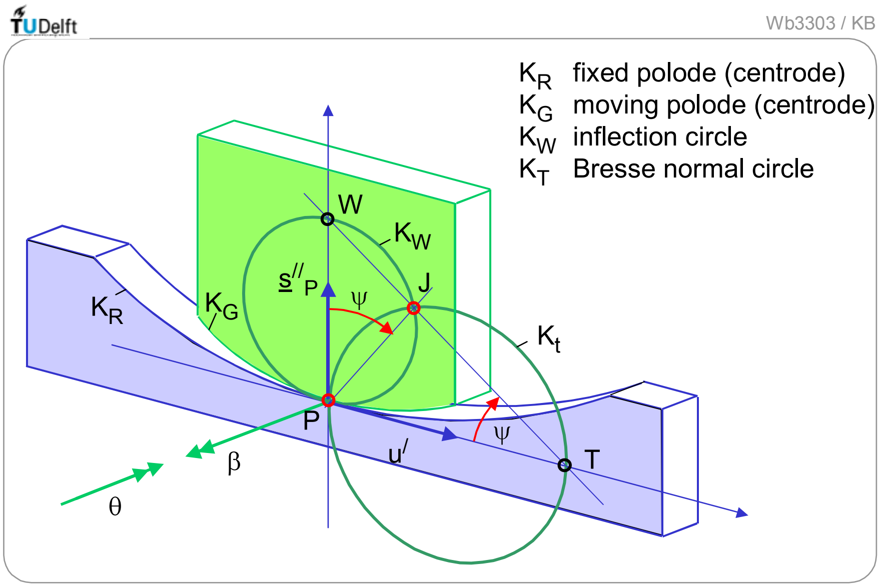

Fig. 4.7.1 Inflection circle and Bresse Normal circle of a moving plane

The first step is to construct the pole tangent using the two points A and B, of which the radii of curvature (the points A_0 and B_0) are known. Now the construction is repeated for the points A and C. This construction must be done however in the reversed order, since point C_0 is unknown. The stepwise sequence of the construction has been depicted in fig. 4.6.3.
The quality of the dwell will not be considered further. It can be remarked that points on the coupler curve that have maximum curvature are favourable, because they have instantaneously four points coinciding with the osculating circle. They promise not only a longer dwell, but the dwell occurs also in a limit position of the output link (in many applications this is also required).

4.7 The Bresse circles

Bresse succeeded to enhance the graphical image of the inflection circle of fig. 4.4.2. First of all he proved that the property of "*having no tangential second order component*" belongs also to a circle, called the *Bresse normal circle* (tangential circle). This circle is similar to the inflection circle: through the first order pole, but now with the pole tangent as the middle line. Of course, by their definition, the other intersection point of both circles is the second order pole (no normal component *and* no tangential component of the second order motion).

Fig. 4.7.1 presents the graphical overview according to Bresse. Any planar motion can be represented by two rolling curves: the fixed polode K_R and the moving polode K_G. These curves are the loci of first order poles in the fixed and the moving plane respectively. Sometimes they can be recognised as materialised parts (for instance in case of a pair of gears), but in general they are just imaginary curves.

The point P attached to the moving plane has its second order motion vector $s^{//}{}_P$ along the pole normal. Because this vector makes angle ψ with pole radius JP according eq.(3.20), this angle can be recognised twice as drawn in the figure.
Point W, called the *inflection centre*, has a special property of its path. In general an inflection point has three points infinitesimally coinciding with the tangent to the path. Because of symmetry the path of point W has four points coinciding with the tangent and has therefor a better straight part (there exists however also a point on the inflection circle with five coinciding points, the point of Ball, which will not be considered here further).

In figure 4.7.1 many useful relations can be detected, for instance:
The first order pole vector (geometric pole velocity) is equal to the first order vector of the inflection centre, and the size is proportional to the diameter of the inflection circle D:

$$s^{/}_{W} = u^{/} = -\beta^{/} \cdot D \tag{4.9}$$

The second order vector $\underline{s}^{//}{}_P$ (P attached to the moving plane) can be expressed as

$$\underline{s}^{//}_{P} = \underline{u}^{/} \times \underline{\beta}^{/} \quad \text{and} \quad s^{//}_{P} = (\beta^{/})^2 \cdot D \tag{4.10}$$

using the sine product of the first order vectors of u and β. Note the direction of rotation vector $\underline{\beta}$, which is in accordance with an orthogonal Cartesian co-ordinate system (the rotation angle θ of the moving plane in fig. 4.4.1 was counted in the opposite direction and consequently vector $u^{/}$ has been drawn in positive pole tangent direction). Actually the diameter D could also be written as:

$$D = \frac{du}{d\theta} = -\frac{u^{/}}{\beta^{/}} \tag{4.11}$$

and this expresses that D depends only on first order motion.

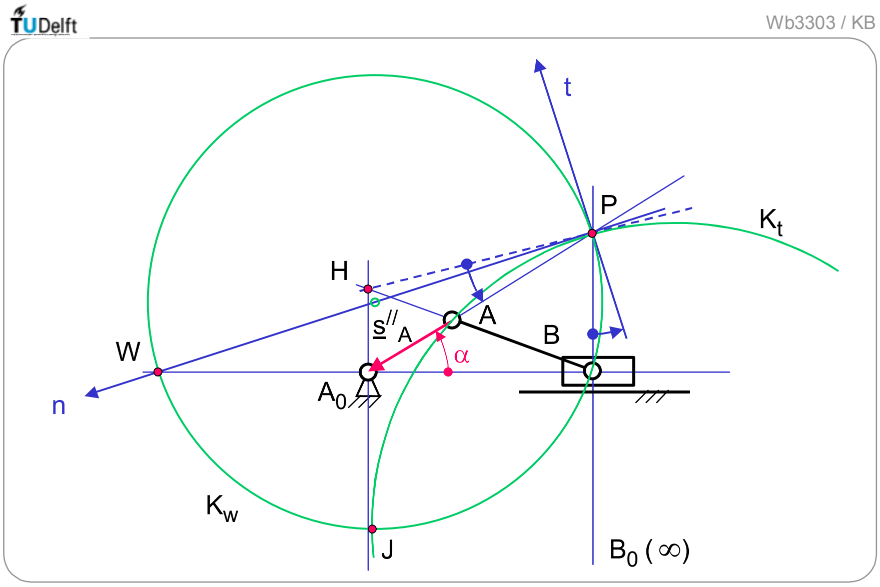

Fig. 4.7.2 *Construction of the 2nd order pole J directly with Bresse's circles (slider-crank mechanism)*

It proves thus also that:

The instantaneous centre of rotation and the inflection circle are time independent.

Angle ψ, used in the definition of the Bresse normal circle, depends on second order motion according eq.(3.20). In case that timed motion is considered, this angle changes also with time. The figure 4.7.1 remains the same, but

The Bresse normal circle and the acceleration pole are time dependent.

Example
The overview presented in figure 4.7.1 offers various possibilities to approach problems of moving planes. To demonstrate this, the slider-crank mechanism will be considered, see fig. 4.7.2, of which the second order pole of the coupler plane is to be found. This can be done very elegantly with the graphical approach.

- Draw the first order pole P and the help pole H for the two points A and B.
- Use the construction of Bobillier to find the pole tangent t (apply $\angle$ HPA to line PB).
- Point B performs a straight line and must therefore lie on the inflection circle.
- Point A has no tangential second order motion along its path (the crank circle), assumed that crank angle α will be the driving angle. So point A must lie on the Bresse normal circle.
- Both circles can be drawn now and their (second) intersection point is the second order pole J.